Advanced Mathematical Economics

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Contents

Chapter 8

Scalar parabolic partial differential equations

8.1 Introduction

Parabolic partial differential equations involve a known function F depending on two independent variables (t,x), an unknown function of them $u(t,x)$, the first partial derivative as regards t and first and second partial derivatives as regards the "spatial" variable x:

$$F(t,x,u,u_t,u_x,u_{xx}) = 0$$

where $u : \mathcal{T} \times \mathrm{X} \to \mathbb{R}$, where $\mathcal{T} \subseteq \mathbb{R}_+$ and $\mathrm{X} \subseteq \mathbb{R}$.

In its simplest form, $F(u_t,u_{xx}) = 0$, the equation models a distribution featuring dispersion through time, for a cross section variable, generated by spatial contact (think about the time distribution of a pollutant spreading within a lake in which the water is completely still). Equation $F(u_t,u_x,u_{xx}) = 0$ features both dispersion and advection behaviors (think about the time distribution of a pollutant spreading within a river). Equation $F(u_t,u,u_x,u_{xx}) = 0$ jointly displays dispersion, advection and growth or decay behaviors (think about a time distribution of a pollutant spreading within a river, in which there is a permanent flow of new pollutants being dumped into the river). The independent terms appear in function $F(.)$ if there are some time or spatial specific components.

We will also see in the next chapter that there is a close connection between partial differential equations and stochastic differential equations. This implied that continuous-time finance has been using parabolic PDE's since the beginning of the 1970's.

In economics and finance applications it is important to distinguish between **forward** (FPDE) and **backward** (BPDE) parabolic PDE's. While the first are complemented with an initial distribution and generate a flow of distributions forward in time, the latter are complemented with a terminal distribution and its solution generate a flow of distributions consistent with that terminal constraint. While for FPDE the terminal distribution is unknown, for BPDE the distribution at time $t = 0$ is unknown. For planar systems, we may have forward, backward or forward-backward (FBPDE) parabolic PDE's. The last case can be seen as a generalization of the saddle-path dynamics for ODE's.

In mathematical finance most applications, such as the Black and Scholes (1973) model, most PDE's are of the backward type. In economics there is recent interest in PDE's related to the topical importance of distribution issues, and, in particular spatial dynamics modelled by BPDE. Optimal control of PDE's and the mean-field games usually lead to FBPDE's.

Again, the body of theory and application of parabolic PDE's is huge. We only present next some very introductory results and applications.

Let $u(t,x)$ where $(t,x) \in \mathcal{T} \times \mathrm{X} \subseteq \mathbb{R} \times \mathbb{R}_+$ is an at least $C^{2,1}(\mathbb{R}_+, \mathbb{R})$ function[1]. We can define

- linear parabolic PDE

$$u_t = a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u + d(t,x)$$

  if $F(.)$ is linear in u and all its derivatives, and the coefficients are independent from u

- a semi-linear parabolic PDE

$$u_t = a(t,x)u_{xx} + b(t,x)u_x + c(x,t,u)$$

  it $F(.)$ is linear in the derivatives of u, and the coefficients are independent from u

- a quasi-linear parabolic PDE

$$u_t = a(x,t,u)u_{xx} + b(x,t,u)u_x + c(x,t,u)$$

  if $F(.)$ is linear in the derivatives of u, but the coefficients can be functions of u.

Consider the simplest linear parabolic equation with constant coefficients, sometimes called the **diffusion equation with advection and growth**,

$$u_t = au_{xx} + bu_x + cu + d.$$

The time-behavior of u depends on three terms: a diffusion term, au_{xx}, a transport term, bu_x, and a growth term $cu + d$. If $a > 0$ ($a < 0$) the equation is sometimes called a **forward** FPDE (**backward** BPDE) equation, because the diffusion operator works forward (backward) in time. The second term introduces a behavior similar to the first-order PDE: it involves a translation of the solution along the direction x. The third term generates a time behavior of the whole distribution $u(x,.)$ in a way similar to a solution of a ordinary differential equation, that is, it involves stability or instability properties.

[1]It is, at least, differentiable to the second order as regards x and to the first order as regards t.

In the case of a parabolic PDE the stability or instability properties are related to the whole distribution: we have **stability in a distributional sense** if there is a solution $u(t,x) = \overline{u}(x)$ such that

$$\lim_{t\to\infty} u(t,x) = \overline{u}(x)$$

where $\overline{u}(x)$ is a stationary distribution.

An important element regarding the existence and characterization of the solution of PDE's is related to the characteristics of the support of the distribution X. We can distinguish between three main cases:

- unbounded or infinite case $\mathrm{X} = (-\infty, \infty)$
- the semi-bounded of semi-infinite case $\mathrm{X} = [0, \infty)$ or $\mathrm{X} = (-\infty, 0]$, where 0 can be substituted by any finite number
- the bounded case $\mathrm{X} = (\underline{x}, \overline{x})$ where both limits are finite.

In order to define **problems involving parabolic PDE's** we have to supplement it with a distribution referred to a point in time (an initial distribution for the forward PDE or terminal distribution for a backward PDE), and possibly conditions involving known values for the values of $u(t,x)$ at the boundaries of X) (so called boundary conditions), i.e, $x \in \partial\mathrm{X}$.

A problem is said to be **well-posed** if there is a solution to the PDE that satisfies jointly the initial (or terminal) and the boundary conditions and it is continuous at those points. In this case we say we have a **classic solution**. If a problem is not well-posed it is **ill-posed**. In this case there are no solutions or classic solutions do not exist (but generalized solutions can exist).

A necessary condition for a problem involving a FPDE to be well posed is that it is supplemented with an initial condition in time, and a necessary condition for a problem BPDE to be well-posed is that it involves a terminal condition in time.

Next we will present the solutions for some simple equations and problems.

8.2 The simplest linear forward equation

8.2.1 The heat equation

The simplest linear parabolic PDE is the heat equation, where $u(t,x)$ and is formalized by the linear forward parabolic PDE

$$u_t - u_{xx} = 0 \tag{8.1}$$

[2]. It describes the dynamics of the temperature distribution when spatial differences in temperature drive the change in spatial distribution of temperature across time. Consider a homogeneous rod with infinite width and let $u(t,x)$ be the temperature at point $x \in (-\infty, \infty)$ at time $t \geq 0$.

[2] The first formulation of the heat equation is attributed to Fourier in a presentation to the Institut de France, and in a book with title *Theorie de la Propagation de la Chaleur dans les Solides* both in 1807.

Consider a small segment of the rod between points x and $x + \Delta x$, where $\Delta x > 0$. The difference in the temperature between the two boundaries of the segment

$$u(t, x + \Delta x) - u(t, x) = \int_x^{x+\Delta x} u_x(t, z)\, dz$$

is a measure of the average temperature in the segment at time t. The instantaneous change in average temperature in the segment is

$$\frac{d}{dt}\Big(\int_x^{x+\Delta x} u(t, z)\, dz \Big) = \int_x^{x+\Delta x} u_t(t, z)\, dz$$

If there is a hotter spot located outside the segment, for instance in a leftward region, and because the heat flows from hot to colder regions, then temperature in the segment Δx is lower then in the leftward region, implying $u_x(t, x) < 0$, and it is higher than in the rightward region, implying $u_x(t, x + \Delta x) < 0$, and the gradient in the leftward boundary is higher in absolute terms that the rightward $u_x(t, x) - u_x(t, x + \Delta x) < 0$. Therefore, the temperature flow is

$$u_x(t, x + \Delta x) - u_x(t, x) = \int_x^{x+\Delta x} u_{xx}\ (t, z)\, dz.$$

If is assumed that the instantaneous change in the segment's temperature is equal to the heat that flows through the segment, then

$$\int_x^{x+\Delta x} u_t(t, z)\, dz = \int_x^{x+\Delta x} u_{xx}\ (t, z)\, dz.$$

which is equivalent to

$$\int_x^{x+\Delta x} u_t(t, z) - u_{xx}\ (t, z)\, dz = 0,$$

which is holds if and only if equation (8.1) is satisfied.

Next we define and solve the simplest linear scalar parabolic partial differential equation $u_t(t, x) = a\, u_{xx}(t, x)$ to address the differences in the solution when we consider the domain of x, X, the existence of side conditions and the sign of a.

We start with the forward equation, where $a > 0$, in subsection 8.2.3 and deal next with the backward equation 8.2.5

8.2.2 Fourier transforms

There are several methods to solve linear parabolic PDE's. When the domain of the independent variable x is $(-\infty, \infty)$, the most direct method to find a solution is by using Fourier and inverse Fourier transforms (see Appendix 8.7).

The method of obtaining a solution follows three steps: first, we transform function $u(t, x)$ such that the PDE is transformed into a parameterized ordinary differential equation; second we solve this ODE; and finally we transform back to the original function. When the domain of x is not the double-infinite we may have to adapt this method.

There are several possible transformations: sine, cosine, Laplace, Mellin or Fourier transforms. Next we use the Fourier transform approach.

The **Fourier transform** of $u(t,x)$, taking t as a parameter, is [3]

$$U(t,\omega) = \mathcal{F}[u(t,x)](\omega) \equiv \int_{-\infty}^{\infty} u(t,x)e^{-2\pi i\omega x}dx \tag{8.2}$$

where $i^2 = -1$ and the **inverse Fourier transform** is

$$u(t,x) = \mathcal{F}^{-1}[U(t,\omega)](x) \equiv \int_{-\infty}^{\infty} U(t,\omega)e^{2\pi i\omega x}d\omega. \tag{8.3}$$

Time derivatives can also have Fourier transform representations: first derivative representations are

$$u_t(t,x) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} U(t,\omega)e^{2\pi i\omega x}d\omega = \int_{-\infty}^{\infty} U_t(t,\omega)e^{2\pi i\omega x}d\omega,$$

and

$$u_x(t,x) = \frac{\partial}{\partial x}\int_{-\infty}^{\infty} U(t,\omega)e^{2\pi i\omega x}d\omega = \int_{-\infty}^{\infty} 2\pi\omega i\, U(t,\omega)e^{2\pi i\omega x}d\omega,$$

and the second derivative is

$$u_{xx}(t,x) = \int_{-\infty}^{\infty} (2\pi\omega i)^2 U(t,\omega)e^{2\pi i\omega x}d\omega = -\int_{-\infty}^{\infty} (2\pi\omega)^2 U(t,\omega)e^{2\pi i\omega x}d\omega.$$

Next we prove the relationship between a convolution of functions and the multiplication of Fourier transforms. The function $u(t,x)$ is a convolution if it can bewritten as

$$u(t,x) = v(t,x) * y(t,x) \equiv \int_{-\infty}^{\infty} v(t,\xi)y(t,x-\xi)d\xi,$$

where $v(t,x)$ and $y(t,x)$ are integrable functions in the domain $\mathbb{R}_+ \times \mathbb{R}$. Let Fourier transform of $u(t,x)$ be written as a product of two Fourier transforms,

$$U(t,\omega) = \mathcal{F}[u(t,x)](\omega) = V(t,\omega)Y(t,\omega)$$

where $V(t,\omega) = \mathcal{F}[v(t,x)](\omega)$ and $Y(t,\omega) = \mathcal{F}[y(t,x)](\omega)$. Then $u(t,x)$ is the inverse Fourier transform of $U(t,\omega)$ if and only if $u(t,x)$ is the convolution

$$u(t,x) = \mathcal{F}^{-1}[U(t,\omega)](x) = \mathcal{F}^{-1}[V(t,\omega)Y(t,\omega)](x) = v(t,x) * y(t,x).$$

8.2.3 The forward heat equation in the infinite domain

In this subsection we solve the slightly more general version of equation (8.1) in the infinite domain for an arbitrary bounded initial condition and for a given initial conditions. The last two are versions of Cauchy problems in which the side conditions refer to $t = 0$.

[3] There are different definitions of Fourier transforms, we use the definition by, v.g., Kammler (2000).

Free but bounded initial condition

The simplest linear PDE for an infinite domain $X = \mathbb{R}$

$$u_t - au_{xx} = 0, \; (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{8.4}$$

where $a > 0$.

Proposition 1. *Let $k(x)$ be an arbitrary but bounded function, i.e. satisfying $\int_{-\infty}^{\infty} |k(x)|dx < \infty$. Then the solution to PDE* (8.4) *is*

$$u(t,x) = \begin{cases} k(x), & (t,x) \in \{t=0\} \; \times \mathbb{R} \\ \dfrac{1}{2\sqrt{\pi a t}} \displaystyle\int_{-\infty}^{\infty} k(\xi) e^{-\frac{(x-\xi)^2}{4at}} \; d\xi, & (t,x) \in \mathbb{R}_{++} \times \mathbb{R} \end{cases} \tag{8.5}$$

Proof. Applying the previous definition of Fourier transform, equation (8.4) becomes

$$u_t - au_{xx} = \int_{-\infty}^{\infty} \left(U_t(t,\omega) + a(2\pi\omega)^2 U(t,\omega)\right) e^{2\pi i\omega x} d\omega = 0.$$

This equation is satisfied if $U(t,\omega)$ is solves

$$U_t(t,\omega) = \lambda(\omega)\, U(t,\omega).$$

where $\lambda(\omega) = -(2\pi\omega)^2 a$. The solution for this ODE is

$$U(t,\omega) = K(\omega)\, G(t,\omega)$$

where $G(.)$ is called the **Gaussian kernel**

$$G(\omega,t) \equiv \begin{cases} 1, & t=0 \\ e^{\lambda(\omega)\, t}, & t>0 \end{cases}$$

and the function $K(\omega)$ is arbitrary. To obtain the solution in terms of the original function, $u(t,x)$, we perform an inverse Fourier transform

$$u(t,x) = \mathcal{F}^{-1}(U(t,\omega)) = \mathcal{F}^{-1}(K(\omega)\, G(t,\omega)) = k(x) * g(t,x)$$

where $k(x) * g(t,x)$ is a convolution, that is

$$k(x) * g(t,x) = \int_{-\infty}^{\infty} k(\xi) g(t, x-\xi) d\xi.$$

Using the tables in the Appendix, for $g(x,t) = \mathcal{F}^{-1}[G(t,\omega)](x)$ the Gaussian kernel in the initial variable is

$$g(t,x) = \begin{cases} \delta(x), & t=0 \\ \dfrac{e^{-\frac{x^2}{4at}}}{2\sqrt{\pi\, a\, t}}, & t>0 \end{cases}$$

where $\delta(.)$ is the Dirac's delta function.

Therefore, because and $k(x) = \mathcal{F}^{-1}[K(t\omega)](x)$,

$$u(t,x) = \begin{cases} k(x), & t = 0 \\ \dfrac{1}{2\sqrt{\pi a t}} \displaystyle\int_{-\infty}^{\infty} k(\xi) e^{-\frac{(x-\xi)^2}{4at}}\, d\xi, & t > 0 \end{cases} \tag{8.6}$$

where $k(x)$ **is an arbitrary but bounded function**, i.e. satisfying $\int_{-\infty}^{\infty} |k(x)| dx < \infty$, and because $\int_{-\infty}^{\infty} k(\xi)\delta(x-\xi)d\xi = k(x)$. □

Two observations can be made concerning the solution of this PDE.

First, applying the Fourier transform, we change from a distribution in the original variables x to a frequency distribution ω.

The transformed PDE becomes a linear ODE the coefficient is eigenfunction

$$\lambda(\omega) = -(2\pi\omega)^2 a$$

which is real and non-positive for any $\omega \in \mathbb{R}$: $\lambda(0) = 0$ and $\lambda(\omega) < 0$ for $\omega \neq 0$ and, $\lim_{\omega\to\pm\infty} \lambda(\omega) = -\infty$. This means that $\lim_{\omega\to\pm\infty} U(t,\omega) = 0$ for any t if $K(\omega)$ is bounded.

Second, associated to the previous property is the solution of $u(t,x)$ is an expected value of the arbitrary function where the density function is a Gaussian density function with average 0 and variance $2\,a\,t$.

Initial value problem Now we consider a well-posed linear FPDE. Assume we know the distribution at time $t = 0$, then we have an **initial value problem**

$$\begin{cases} u_t = a u_{xx}, & (t,x) \in (0,\infty) \times (-\infty,\infty) \\ u(0,x) = \phi(x) & (t,x) \in \{t = 0\} \times (-\infty,\infty) \end{cases} \tag{8.7}$$

where $a > 0$ and $\phi(x)$ is a known bounded function. Applying (8.6), the solution is

$$u(t,x) = \begin{cases} \phi(x), & (t,x) \in \{t=0\} \times \mathbb{R} \\ \displaystyle\int_{-\infty}^{\infty} \frac{\phi(\xi)}{2\sqrt{\pi\,a\,t}}\, e^{-\dfrac{(x-\xi)^2}{4at}}\, d\xi, & (t,x) \in \mathbb{R}_{++} \times \mathbb{R} \end{cases}$$

because $\int_{-\infty}^{\infty} \phi(\xi)\delta(x-\xi)d\xi = \phi(x)$.

Example Figure 8.1 illustrates the behavior of the solution for $a = 1$ and $\phi(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$, which is simplified to

$$u(t,x) = \frac{1}{\sqrt{\pi(1+4\,t)}}\, e^{-\dfrac{x^2}{1+4\,t}}.$$

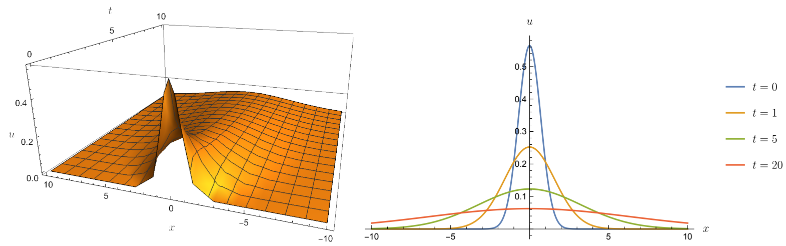

Figure 8.1: Solution for the initial value problem for the heat equation with $a = 1$ and $\phi(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$.

As can be seen, the solution decays through time and converges to a homogeneous distribution

$$\lim_{t\to\infty} u(t,x) = 0, \forall x \in (-\infty, \infty)$$

However, a conservation law holds,

$$\int_{-\infty}^{\infty} u(t,x)\, dx = 1, \text{for each}\, t \geq 0$$

Piecewise-constant initial condition We consider the heat equation with the initial condition

$$\phi(x) = \begin{cases} \phi_0, & \text{if } x \in [\underline{x}, \overline{x}] \\ 0 & \text{if } x \notin [\underline{x}, \overline{x}) \end{cases}$$

where $\underline{x} < \overline{x}$ are both finite. In this case, the solution to the problem is

$$u(t,x) = \phi_0 \left[\, \Phi\left(\frac{x - \underline{x}}{\sqrt{2at}} \right) - \Phi\left(\frac{x - \overline{x}}{\sqrt{2at}} \right) \right] \tag{8.8}$$

where $\Phi(z)$ is the standard normal distribution function

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{z^2}{2}} dz \in (0,1).$$

Observe that $\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = (2\pi)^{\frac{1}{2}}$.

The solution of equation (8.8) is illustrated in Figure 8.2.

In order to prove this result, applying the general solution in equation (8.6) yields the solution of the initial-value problem

$$u(t,x) = \frac{\phi_0}{2\sqrt{\pi a t}} \int_{\underline{x}}^{\overline{x}} e^{-\frac{(x-\xi)^2}{4at}} d\xi.$$

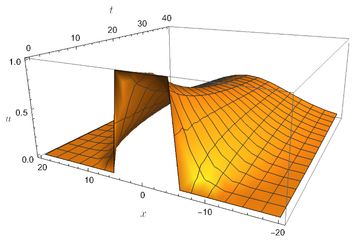

Figure 8.2: Solution for the initial value problem for the heat equation with $a = 1$ and piecewise initial condition.

To simplify the expression, we make the transformation of variables $z \equiv (x-\xi)/\sqrt{2at}$, and denote $\overline{z} \equiv (\overline{x}-\xi)/\sqrt{2at}$ and $\underline{z} \equiv (\underline{x}-\xi)/\sqrt{2at}$. Then, because $dz = -1/\sqrt{2at}d\xi$ the solution simplifies [4]

$$\begin{aligned}
\frac{1}{\sqrt{4\pi at}}\int_{\underline{x}}^{\overline{x}} e^{-(x-\xi)^2/4at}d\xi &= -\frac{\sqrt{2at}}{\sqrt{4\pi at}}\int_{(x-\underline{x})/\sqrt{2at}}^{(x-\overline{x})/\sqrt{2at}} e^{-z^2/2}dz \\
&= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{(x-\underline{x})/\sqrt{2at}} e^{-z^2/2}dz - \int_{-\infty}^{(x-\overline{x})/\sqrt{2at}} e^{-z^2/2}dz\right) = \\
&= \Phi\left(\frac{x-\underline{x}}{\sqrt{2at}}\right) - \Phi\left(\frac{x-\overline{x}}{\sqrt{2at}}\right).
\end{aligned}$$

8.2.4 The forward linear equation in the semi-infinite domain

Now consider the equation defined on the semi-infinite domain for x, that is $\mathrm{X} = \mathbb{R}_+$. This case is more interesting for economic applications in which the independent variable can only take non-negative values, for instance when x refers to a stock.

The FPDE we consider is

$$u_t - a u_{xx} = 0,\ (t,x) \in \mathbb{R}^2_+ \tag{8.9}$$

where $a > 0$.

Proposition 2. *The solution to equation* (8.11) *is*

$$u(t,x) = \frac{1}{2\sqrt{\pi\, a\, t}}\int_0^\infty k\,(\xi)\left(e^{-\frac{(x-\xi)^2}{4\,a\,t}} - e^{-\frac{(x+\xi)^2}{4\,a\,t}}\right)\,d\xi,\ t > 0. \tag{8.10}$$

where $k(x): X \to \mathbb{R}$ is an arbitrary bounded function.

[4] Recalling the formula for integration by substitution of variables, if we set $z = \varphi(\xi)$ and $\xi \in (a,b)$ then

$$\int_{\varphi(a)}^{\varphi(b)} f(z)dz = \int_a^b f(\varphi(\xi))\,\varphi'(\xi)\,d\xi.$$

Proof. We solve this equation by using the **method of images**. It consists in introducing the following extension to the arbitrary function $k(x)$

$$\tilde{k}(x) = \begin{cases} k(x), & \text{if } x \geq 0 \\ -k(-x) & \text{if } x < 0 \end{cases}$$

where $k(.)$ is an odd function satisfying $k(-x) = -k(x)$. Using the solution (8.6) for $t > 0$ we have

$$\begin{aligned} u(t,x) &= \frac{1}{2\sqrt{\pi a t}} \int_{-\infty}^{\infty} \tilde{k}\ (\xi) e^{-\frac{(x-\xi)^2}{4at}}\ d\xi = \\ &= \frac{1}{2\sqrt{\pi a t}} \left(\int_{-\infty}^{0} \tilde{k}\ (\xi) e^{-\frac{(x-\xi)^2}{4at}}\ d\xi + \int_{0}^{\infty} \tilde{k}\ (\xi) e^{-\frac{(x-\xi)^2}{4at}}\ d\xi \right) = \\ &= \frac{1}{2\sqrt{\pi a t}} \left(-\int_{0}^{\infty} k\ (\xi) e^{-\frac{(x+\xi)^2}{4at}}\ d\xi + \int_{0}^{\infty} k\ (\xi) e^{-\frac{(x-\xi)^2}{4at}}\ d\xi \right) \end{aligned}$$

where the last step involves integration by substitution: i.e., if we define $u = -x$ for $x \in [0, \infty)$ then $\int_{-\infty}^{0} f(u)du = -\int_{\infty}^{0} f(-x)dx = \int_{0}^{\infty} f(-x)dx$. Then the solution of equation (8.11) is equation (8.11). □

The solution to the initial-value problem

$$\begin{cases} u_t - a u_{xx} = 0, & \text{for } a > 0,\ (t,x) \in \mathbb{R}_+^2 \\ u(0,t) = u_0(x), & \text{for } (t,x) \in \{t=0\} \times \mathbb{R}_+ \end{cases} \tag{8.11}$$

is

$$u(t,x) = \frac{1}{2\sqrt{\pi\, a\, t}} \int_0^{\infty} u_0(\xi) \left(e^{-\frac{(x-\xi)^2}{4\,a\,t}} - e^{-\frac{(x+\xi)^2}{4\,a\,t}} \right)\ d\xi,\ t > 0. \tag{8.12}$$

We obtain this result by direct application of equation (8.12).

Example Consider the initial-value problem in which the initial distribution is log-normal

$$u_0(x) = \frac{e^{-\dfrac{(\ln x\ - \mu)^2}{2\sigma^2}}}{2\sqrt{\pi\, x^2\, \sigma^2}}$$

if we substitute in equation (8.12) we have the solution depicted in Figure 8.4 for several moments in time.

We observe that the solution is not conservative, i.e. the integral $U(t) = \int_0^{\infty} u(t,x)\, dx$ decays in time such that $\lim_{t\to\infty}\ U(t) = 0$.

8.2.5 The backward heat equation

In finance applications and associated to Euler equation in optimal control problems, we sometimes need to solve backward parabolic PDE.

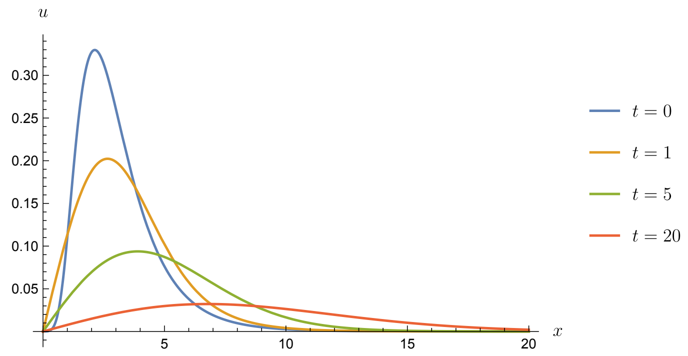

Figure 8.3: Solution for the initial value problem for the heat equation in the semi-infinite line with $a = 1$ and an initial log-normal density.

The simplest parabolic BPDE equation in the infinite domain for x and for the semi-infinite domain for t is

$$u_t + au_{xx} = 0, \; (t, x) \in [0, T] \times (-\infty, \infty) \tag{8.13}$$

where $a > 0$.

Proposition 3. *Consider the BPDE equation* (8.13)*. The solution is*

$$u(t,x) = \begin{cases} k(x), & t = T \\ \frac{1}{\sqrt{4\pi a(T-t)}} \int_{-\infty}^{\infty} k(\xi) e^{-\frac{(x-\xi)^2}{4a(T-t)}} \, d\xi, & t \in (0, T) \end{cases}$$

Proof. In order to solve it we introduce a change in variables $\tau = T - t$ and consider a change in the variable $v(\tau, x) = u(t(\tau), x)$ where $t(\tau) = T - \tau$. As

$$v_\tau(\tau, x) = -u_t(t(\tau), x), \text{ and } v_{xx}(\tau, x) = u_{xx}(t(\tau), x)$$

Then $u_t(t, x) = -au_{xx}(t, x)$ is equivalent to

$$v_\tau(\tau, x) = av_{xx}(\tau, x).$$

Using the solution already found in equation (8.6) we get

$$v(\tau, x) = \begin{cases} k(x), & \tau = 0 \\ \int_{-\infty}^{\infty} k(\xi) \left(4\pi a\tau\right)^{-1/2} e^{-\frac{(x-\xi)^2}{4a\tau}} \, d\xi, & \tau \in (0, T). \end{cases}$$

Transforming back to $u(t, x)$ we have solution. □

A problem involving a backward PDE is only well posed if together with the PDE we have a terminal condition, for example $u(T, x) = \phi_T(x)$. In this case the value of the variable at time $t = 0$ becomes endogenous.

Consider the **terminal-value problem**

$$\begin{cases} u_t = -a u_{xx}, & (t,x) \in (-\infty,\infty) \times (0,T] \\ u(T,x) = \phi_T(x) & (t,x) \in (-\infty,\infty) \times \{\ t=T\}. \end{cases}$$

The solution is

$$u(t,x) = \begin{cases} \phi_T(x), & (t,x) \in \{t=T\}\ \times \mathbb{R} \\ \dfrac{1}{\sqrt{4\pi a(T-t)}} \displaystyle\int_{-\infty}^{\infty} \phi_T(\xi) e^{-\frac{(x-\xi)^2}{4a(T-t)}}\ d\xi, & (t,x) \in (0,T) \times \mathbb{R} \end{cases}$$

The initial distribution can be obtained by setting $t=0$

$$u(0,T) = \frac{1}{\sqrt{4\pi a(T-t)}} \int_{-\infty}^{\infty} \phi_T(\xi) e^{-\frac{(x-\xi)^2}{4aT}}\ d\xi.$$

8.3 The homogeneous linear PDE

The general forward linear parabolic PDE in the infinite domain is

$$u_t = a\, u_{xx} + b\, u_x + c\, u,\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}$$

where $a > 0$. The dynamics of $u(t,x)$ contains three terms: a diffusion term, if $a \neq 0$, a transport term, if $b \neq 0$, and a growth or decay term if $c > 0$ or $c < 0$.

In order to solve the equation, we can follow one of two alternative methods:

1. transform the equation into a heat equation, solve the heat equation and transform back to the initial variables.

2. apply the Fourier transform method to transform the PDE into a parameterized ODE, solve it, and apply inverse Fourier transforms.

8.3.1 Equation without transport term

If the linear forward PDE does not contain a transport term, we have

$$u_t = a\, u_{xx} + c\, u,\ (t,x) \in (0,\infty) \times (-\infty,\infty) \tag{8.14}$$

where $a > 0$ and $c \neq 0$, which has solution, for an arbitrary bounded function $\phi(x)$

$$u(t,x) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \phi(\xi)\delta(x-\xi) d\xi = \phi(x), & (t,x) \in \{t=0\}\ \times \mathbb{R} \\ e^{c\,t} \displaystyle\int_{-\infty}^{\infty} \phi(\xi) \frac{1}{\sqrt{4\,\pi\,a\,t}} e^{-\frac{(x-\xi)^2}{4\,a\,t}}\ d\xi, & (t,x) \in \mathbb{R}_+\ \times \mathbb{R}. \end{cases}$$

To find the solution, we use the first method. First, define $v(t,x) = e^{-c\,t}u(t,x)$, which has derivatives $v_t = -ce^{-c\,t}u + e^{-c\,t}u_t$ and $v_{xx} = e^{-ct}u_{xx}$. Second, equation (8.14) is equivalent to the simplest linear equation $v_t = av_{xx}$ which has solution (8.6). Third, as $u(t,x) = e^{c\,t}\,v(t,x)$ we obtain the solution

The dynamics of the solution depends crucially on the sign of b:

$$\lim_{t\to\infty} u(t,x) = \begin{cases} 0 & \text{if } c < 0 \\ \infty & \text{if } c > 0 \end{cases}$$

Figure 8.4 illustrates the cases in which $c < 0$ and $c > 0$. In both cases we see that the long-time behavior of the solution is commanded by $e^{c\,t}$: if $c < 0$ then $\lim_{t\to\infty} u(t,x) = 0$, for any $x \in \mathbb{R}$, and if $c > 0$ then $\lim_{t\to\infty} u(t,x) \propto \lim_{t\to\infty} e^{c\,t} = \infty$, for any $x \in \mathbb{R}$.

This means that the diffusion equation display **asymptotic stability** if $c < 0$ and **instability** if $c > 0$, both in a distributional sense. In the first case the solution $u(t,x)$ is bounded and in the second case it is unbounded.

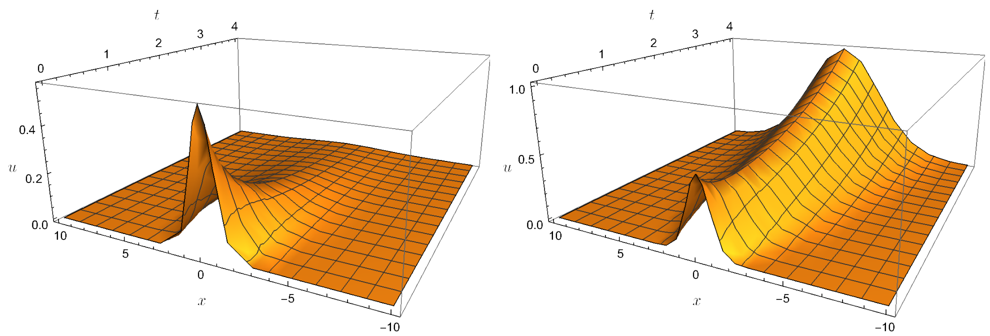

Figure 8.4: Solution for the initial value problem for the heat equation with $a = 1$ and $\phi(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$ and $c = -0.5$ and $c = 0.5$.

8.3.2 The general homogeneous diffusion equation

The initial value problem for a a general linear homogeneous (forward) diffusion equation is

$$\begin{cases} u_t(t,x) = a\,u_{xx}(t,x) + b\,u_x(t,x) + c\,u(t,x), & (t,x) \in (0,\infty) \times (-\infty,\infty) \\ u(0,x) = \phi(x), & (t,x) \in \{t = 0\} \times (-\infty,\infty), \end{cases} \tag{8.15}$$

where $a > 0$, $b \neq 0$ and $c \neq 0$ and $\phi(x)$ is a bounded function over $X = \mathbb{R}$.

Proposition 4. *The solution to problem* (8.15) *is*

$$u(t,x) = \int_{-\infty}^{\infty} \phi(s)\,\frac{1}{\sqrt{4\pi at}}\exp\left(-\frac{(x-s)^2 + 2\,b\,(x-s)\,t + (b^2 - 4ac)\,t^2}{4\,a\,t}\right) ds \tag{8.16}$$

Proof. We will solve this problem using the Fourier transform representation of equation $u_t - (au_{xx} + bu_x + cu) = 0$. Using inverse Fourier transforms yields

$$u_t(t,x) - au_{xx}(t,x) - bu_x - cu(t,x) = \int_{-\infty}^{\infty} e^{2\pi i\omega x}\left[\, U_t(t,\omega) + \lambda(\omega)U(t,\omega)\right]\, d\omega = 0.$$

where the coefficient is a complex-valued function of ω [5]

$$\lambda(\omega) \equiv a(2\pi\omega)^2 - c - 2\pi\, b\, \omega\, i,\ i \equiv \sqrt{-1}$$

Therefore, the PDE (8.15) is equivalent to the linear ODE parameterized by ω

$$U_t(t,\omega) = -\lambda(\omega)U(t,\omega),\ (t,\omega) \in \mathbb{R}_+ \times \mathbb{R},$$

which has the explicit solution

$$U(t,\omega) = \Phi(\omega)\, G(t,\omega),\ \text{for} t \in [0,\infty)$$

where $\Phi(\omega) = \mathcal{F}[\,\phi(x)]\,(\omega)$ is the Fourier transform of the initial distribution, and $G(t,\omega)$ is the Gaussian kernel

$$G(t,\omega) = e^{-\lambda(\omega)t},\ \text{for } t > 0.$$

We obtain the solution of problem (8.15) by applying the inverse Fourier transform

$$u(t,x) = \mathcal{F}^{-1}\left[U(t,\omega)\right]\,(x) = \mathcal{F}^{-1}\left[\Phi(\omega)\, G(t,\omega)\right]\,(x) = \int_{-\infty}^{\infty} \phi(s)g(t,x-s)ds$$

where (see the Appendix 8.7)

$$g(t,y) = \mathcal{F}^{-1}\left[\, e^{-\lambda(\omega)t}\right] \;=\; \frac{1}{\sqrt{4\pi a t}}\exp\left(-\frac{y^2 + 2bty + (b^2 - 4ac)t^2}{4at}\right), \tag{8.17}$$

because $at > 0$. □

Figure 8.5 illustrates the solution (8.16) for negative (left figures) and positive (right figures) values for b and negative (upper figures) and positive (lower figures) values of c. It is clear that while b introduces a transportation in the positive direction, if $b < 0$, or in the negative direction, if $b > 0$, c is associated to the time stability of the whole distribution.

8.4 Non-autonomous linear equation

Next we consider two non-autonomous equations in which there is one term depending on the independent variables (t,x)

[5] The advection term, involving the first derivative has a complex-valued the Fourier transform representation

$$u_x(t,x) = \frac{\partial}{\partial x}\left(\int_{-\infty}^{\infty} U(t,\omega)e^{2\pi i x\omega}d\omega\right) = \int_{-\infty}^{\infty} 2\pi\omega i\, U(t,\omega)e^{2\pi i x\omega}d\omega.$$

.

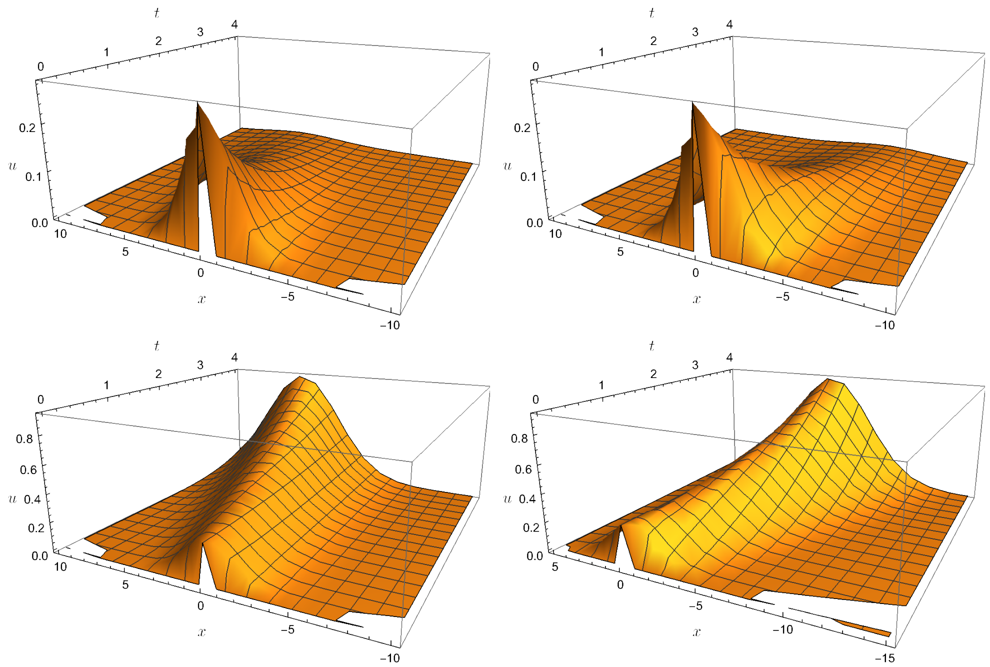

Figure 8.5: Solution for the initial value problem linear PDE for $a = 1$, $b = 1$ and $b = -1$ and $c = -0.5$ and $b = 1$ and $b = -1$ and $c = 0.5$.

Non-homogeneous heat equation The non-homogeneous (forward) heat equation

$$u_t - au_{xx} - b(t,x) = 0, \ (t,x) \in (0,\infty) \times (-\infty,\infty) \tag{8.18}$$

this equation has a component which is not affected by u, although it changes with (t,x).

In order to solve it, we again use inverse Fourier transforms to get an equivalent ODE in transformed variables $U(t,\omega)$,

$$U(t,\omega) = -\lambda(\omega)U(t,\omega) + B(t,\omega)$$

where $B(t,\omega) = \mathcal{F}[b(t,x)](\omega)$ and $\lambda(\omega) = (2\,\pi\,\omega)^2\,a$. The solution to equation (8.18)is

$$U(t,\omega) = K(\omega)G(t,\omega) + \int_0^t B(s,\omega)\,G(t-s,\omega)ds,$$

where $G(t,\cdot)$ is a Gaussian kernel. Applying inverse Fourier transforms yields

$$u(t,x) = k(x) * g(t,x) + \int_0^t b(s,x) * g(t-s,x)ds.$$

Therefore, the solution to the parabolic PDE (8.18) is, for $t > 0$,

$$u(t,x) = \frac{1}{\sqrt{4\pi at}} \int_{-\infty}^{\infty} k(\xi)e^{-\frac{(x-\xi)^2}{4at}}d\xi + \int_0^t \frac{1}{\sqrt{4\pi a(t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4a(t-s)}} b(s,\xi)d\xi ds.$$

The solution can converge to a spatially non-homogenous distribution.

Non-autonomous diffusion equation

Consider the equation

$$u_t = au_{xx} + bu + d(x),\ (t,x) \in (-\infty,\infty) \times (0,\infty)$$

where $a > 0$. It can be proved (see Exercise 1) that the solution for $t > 0$ is

$$u(t,x) = \frac{e^{bt}}{\sqrt{4\pi at}} \int_{-\infty}^{\infty} \phi(\xi) e^{-\frac{(x-\xi)^2}{4at}}\ d\xi + \frac{1}{\sqrt{4\pi a(t-\tau)}}\ \int_0^t\ e^{b(t-\tau)} \int_{-\infty}^{\infty} d(\xi) e^{-\frac{(x-\xi)^2}{4a(t-\tau)}}\ d\xi d\tau$$

8.5 Fokker-Planck-Kolmogorov equation

We will see in the chapter on stochastic differential equations, that the probability distribution of a diffusion process follows a particular parabolic PDE, called the Fokker-Planck-Kolmogorov equation. This equation is having an increase attention, also in economics, as a model for processes satisfying a conservation law as

$$\int_X p(t,x)dx = 1,\ \text{for every}\ \ t \in T$$

where $p(t,x) : T \times X \to (0,1)$.

The Fokker-Planck-Kolmogorov equation is

$$\partial_t p(t,x) = \frac{1}{2}\partial_{xx}\ \Big(a(t,x)^2\, p(t,x)\Big) - \partial_x\Big(b(t,x)\, p(t,x)\Big) \tag{8.19}$$

where we assume $p(0,x)$ is known and satisfies

$$\int_X p(0,x)dx = 1.$$

In applications resulting from stochastic differential equations, the initial state is known $x = x_0$ and the dynamics of a probability distribution is given by Kolmogorov forward equation (or Fokker-Planck equation) and the initial condition $p(0,x) = \delta(x - x_0)$ where $\delta(\cdot)$ is Dirac's delta generalized function.

8.5.1 The simplest problem

The simplest model has constant coefficients $b(t,x) = \mu$ and $a(t,x) = \sigma$ and a Dirac delta initial function:

$$\begin{cases} \partial_t p(t,x) = \dfrac{\sigma^2}{2}\ \partial_{xx} p(t,x) - \mu \partial_x p(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ p(0,x) = \delta(x - x_0) & (t,x) \in \{\ t = 0\}\ \times \mathbb{R} \end{cases} \tag{8.20}$$

The solution is a Gamma probability density

$$p(t,x) = \frac{1}{\sqrt{2\,\pi\,\sigma^2\,t}}\exp -\Big(\frac{x-x_0-\mu\,t}{2\,\sigma^2\,t}\Big)^2$$
$$= \Gamma\big(-\mu t; \frac{\sigma^2}{2}, x-x_0\big)\ (t,x)\in\mathbb{R}_+\times\mathbb{R}$$

Figure 8.6 presents an illustration of the solution

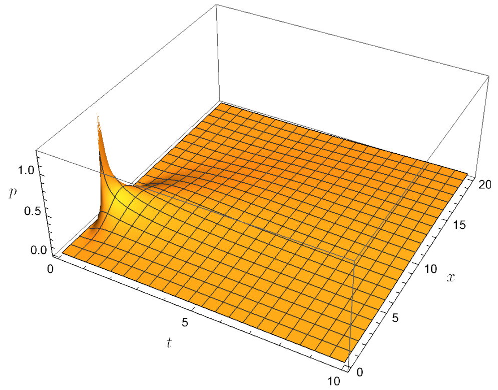

Figure 8.6: Solution for (8.20) for $x_0 = 5$, $\mu = 1$ and $\sigma = 0.5$.

8.5.2 The distribution associated to the Ornstein Uhlenbeck equation

The simplest model has constant coefficients $b(t,x) = \mu_0 + \mu_1\,x$ and $a(t,x) = \sigma$ and a Dirac delta initial function:

$$\begin{cases}\partial_t p(t,x) = \dfrac{\sigma^2}{2}\,\partial_{xx}p(t,x) - (\mu_0+\mu_1\,x)\,\partial_x p(t,x) - \mu_1 p(t,x), & (t,x)\in\mathbb{R}_+\times\mathbb{R}\\ p(0,x) = \delta(x-x_0) & (t,x)\in\{\,t=0\}\ \times\mathbb{R}\end{cases} \tag{8.21}$$

The solution is a Normal density function

$$p(t,x) = \frac{1}{\sqrt{2\,\pi\,\sigma^2\,(1-e^{2\,\mu_1\,t}\big)}}\exp\left\{-\frac{\Big(x-x_0\,e^{\mu_1 t}-\mu_0(1-e^{\mu_1 t})\Big)^2}{2\,\sigma^2\,(1-e^{2\mu_1 t}\ )}\right\}$$
$$= N\Big(x_0 e^{\mu t}\ + \mu_0(1-e^{\mu_1 t}), \frac{\sigma^2}{2}\,(1-e^{2\mu_1 t}\ )\Big)\ (t,x)\in\mathbb{R}_+\times\mathbb{R}$$

Exercise: prove this.

We see that if $\mu_1 < 0$ then

$$\lim_{t\to\infty} p(t,x) \sim N(\mu_0, \frac{\sigma^2}{2})$$

this means that the distribution is ergodic: for any initial value x_0 it tends asymptotically to a normal distribution (see Figure 8.7).

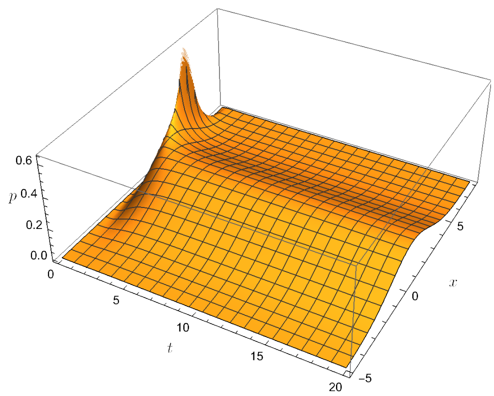

Figure 8.7: Solution for (8.20) for $x_0 = 5$, $\mu_0 = 1$, $\mu_1 = -1$ and $\sigma = 0.5$.

8.6 Economic applications

8.6.1 The distributional Solow model

In Brito (2004) we prove that in an economy in which the capital stock is distributed in an heterogeneous way between regions, $K(t,x)$, if there is an infinite support, and there are free capital flows between regions, the budget constraint for the location x can be represented by the parabolic PDE.

Consider the accounting balance between savings and internal and external investment for a region x at time t

$$I(t,x) + T(t,x) = S(t,x)$$

where $I(t,x)$ and $S(t,x)$ is investment and domestic savings of location x at time t and $T(t,x)$ is the savings flowing to other regions.

Assume that the capital flow for a region of length Δx is symmetric to the capital distribution gradient to neighboring regions:

$$T(t,x)\Delta x = -\left(K_x(x+\Delta x,t) - K_x(t,x)\right)$$

that is capital flows proportionaly and in a reverse direction to the "spatial gradient" of the capital distribution: regions with high capital intensity will tend to "leak" capital to other regions. If $\Delta x \to 0$ leads to $T(t,x) = -K_{xx}(t,x)$.

If there is no depreciation then $I(t,x) = K_t(t,x)$. If the technology is AK and the savings rate is determined as in the Solow model then $S(t,x) = sAK(t,x)$ where $0 < s < 1$ and A is assume to be spatially homogeneous.

Therefore we obtain a distributional Solow equation for an economy composed by heterogenous regions

$$K_t = K_{xx} + sAK, (t,x) \in (-\infty,\infty) \times (0,\infty)$$

We can define a spatially-homogenous balanced growth path (BGP) as

$$\overline{K}(t) = \overline{K}e^{\gamma t}$$

where $\gamma = sA$.

Then, if we define the deviations as regards the BGP as $k(t, x) = K(t, x)e^{-\gamma t}$, we observe that the transitional dynamics is given by the solution of the equation

$$k_t = k_{xx}$$

which is the heat equation. Therefore, given the initial distribution of the capital stock $K(x, 0) = k_0(x)$ the solution for this spatial AK model is

$$K(t,x) = \begin{cases} k_0(x), & t = 0 \\ e^{\gamma t} \int_{-\infty}^{\infty} k_0(\xi) \, (4\pi t)^{-1/2} \, e^{-\frac{(x-\xi)^2}{4t}} \, d\xi, & t > 0 \end{cases}$$

and the solution is similar to the case depicted in Figure 8.2 when $b > 0$.

The main conclusion is that: (1) there is long run growth; (2) , if there are homogenous technologies and preferences the asymptotic distribution will become spatially homogeneous. That is: the so-called β- and σ- convergences can be made consistent !

8.6.2 The option pricing model

The Black and Scholes (1973) model is a case in which a research paper had an immense impact on the operation of the economy. It is related to the onset of derivative markets and basically gave birth to stochastic finance[6].

It provides a formula (the so called Black-Scholes formula) for the value of an European call option when there are two assets, a riskless asset with interest rate r and a underlying asset whose price, S which follows a diffusion process (in a stochastic sense): $dS = \mu S dt + \sigma S dB$ where dB is the standard Brownian motion (see next chapter). An European call option offers the right to buy the underlying asset at time T for a price K fixed at time $t = 0$, which is conventioned to be the moment of the contract.

Under the assumption that there are no arbitrage opportunities Black and Scholes (1973) proved that the price of the option $V = V(t, S)$ is a a function of time, $t \in (0, T)$ and the price of an underlying asset $S \in (0, \infty)$ follows the backward parabolic PDE and has a terminal constraint

$$\begin{cases} V_t(t,S) & = -\frac{\sigma^2 S^2}{2} \, V_{SS}(t,S) - rSV_S(t,S) + rV(t,S), \; (t,S) \in [0,T] \times (0,\infty) \\ V(T,S) & = \max\{ \, S - K, 0 \} \; , \; (t,S) \in \{t = T\} \times (0,\infty). \end{cases} \tag{8.22}$$

[6]Myron Scholes was awarded the Nobel prize in 1997, together with Robert Merton another important contributer to stochastic finance, precisely for this formula. Fisher Black was deceased at the time.

The first equation is valid for any financial option having the same underlying asset dynamics, and the terminal constraint is characteristic of the European call option: because the writer sells the right, but not the obligation, to purchase the underlying asset at the price K at time $t = T$, the buyer is only interested in that purchase if he can sell it at the prevailing market price $S = S(T)$ when that price is higher than the exercise price K. In this case the payoff will be $S(T) - K$. Otherwise he will not execute the option and the terminal payoff will be zero.

The two boundary constraints

$$V(t, 0) = 0, \quad (t, S) \in [0, T] \times \{S = 0\}$$
$$\lim_{S\to\infty} V(t, S) = S, \quad (t, S) \in [0, T] \times \{S \to \infty\},$$

are sometimes referred to, but they are redundant.

The same structure occurs in the Merton's model (see Merton (1974)) which is a seminal paper on the pricing of default bonds. It was the first model on the so-called structural approach to modelling credit risk which is on the foundation of the credit risk models used by rating agencies (see Duffie and Singleton (2003)). In essence, this model assumes that the value of the firm follows a linear diffusion process and it consideres the issuance of a bond with an expiring date T whose indenture gives it absolute priority on the value of the firm at the expiry date. This means that either if the value of the firm is smaller that the face value of the bond the creditor takes possession of the firm and in the opposite case it recovers the face value. In this case, we can interpret the position of the equity owner as holding an European call option over the value of the firm with strike price equal to the face value of the debt and the creditor as having an European put option security.

The price of the European call option [7], given the former assumptions is given by

$$V(t, S) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \quad t \in [0, T] \tag{8.23}$$

where $\Phi(.)$ is cumulative Gaussian density function such that $\Phi(d) = \mathbb{P}(x \le d)$ where

$$d_1 = \frac{\ln(S/K) + (T-t)\left(r + \dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} \tag{8.24}$$

$$d_2 = \frac{\ln(S/K) + (T-t)\left(r - \dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} \tag{8.25}$$

Proof. In order to solve the B-S PDE, which is a non-linear backward parabolic PDE, we transform it to to a quasi-linear parabolic forward PDE, by applying the transformations: $t(\tau) = T - \tau$ and $S = Ke^x$ and setting $u(\tau, x) = V(t(\tau), S(x))$. We can transform the option-pricing problem to the

[7] For the credit risk model S would be the value of assets of a firm, K would be the face value of loan, and T the term of the loan.

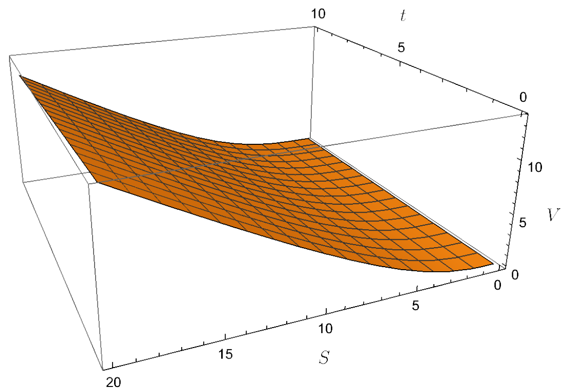

Figure 8.8: Solution for the Black and Scholes model, for $r = 0.02$, $T = 20$, $\sigma = 0.2$, and $K = 10$.

equivalent initial-value problem PDE equivalent to (8.22)

$$\begin{cases} u_\tau = \dfrac{\sigma^2}{2} u_{xx} + \left(r - \dfrac{\sigma^2}{2}\right) u_x - ru, & (\tau, x) \in [0, T] \times (-\infty, \infty) \\ u(0, x) = u_0(x) \end{cases} \tag{8.26}$$

where

$$u_0(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ K\left(e^x - 1\right), & \text{if } x > 0 \end{cases}$$

The PDE is a particular example of equation (8.15), which implies that the solution is

$$\begin{aligned} u(\tau, x) &= \int_{-\infty}^{0} 0\, g(\tau, x - s) ds + K \int_0^\infty \left(e^s - 1\right) g(\tau, x - s) ds \\ &= \frac{K}{\sqrt{2\pi\sigma^2\tau}} \int_0^\infty \left(e^s - 1\right) e^{h(\tau, x - s)} ds \end{aligned}$$

where (from equation (8.17))

$$h(\tau, y) \equiv -\frac{y^2 + 2\tau\left(r - \dfrac{\sigma^2}{2}\right) y + \left(r + \dfrac{\sigma^2}{2}\right)^2 \tau^2}{2\tau\sigma^2}.$$

Then

$$\begin{aligned} u(\tau, x) &= \frac{K}{\sqrt{2\pi\sigma^2\tau}} \left(\int_0^\infty e^{s + h(\tau, x - s)} ds - \int_0^\infty e^{h(\tau, x - s)} ds\right) \\ &= \frac{K}{\sqrt{2\pi\sigma^2\tau}} \left(I_1 - I_2\right). \end{aligned}$$

In order to simplify the integrals it is useful to remember the forms of the error function, $\text{erf}(x)$, and of the Gaussian cumulative distribution $\Phi(x)$,

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-z^2} dz, \ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2} z^2} dz$$

which are related as

$$\Phi(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}} \right)\right].$$

After some algebra we obtain

$$\begin{aligned} s + h(\tau, x-s) &= x - \frac{1}{2}(\delta_1(s))^2 \\ h(\tau, x-s) &= -r\tau - \frac{1}{2}(\delta_2(s))^2 \end{aligned}$$

where

$$\delta_1(s) \equiv \frac{x - s + \left(r + \dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}, \text{ and } \delta_2(s) \equiv \frac{x - s + \left(r - \dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}.$$

Then [8]

$$\begin{aligned} I_1 &= e^x \int_0^{\infty} e^{-\frac{1}{2}(\delta_1(s))^2} ds = \\ &= -\sigma\sqrt{\tau}e^x \int_{d_1}^{-\infty} e^{-\frac{1}{2}\delta_1^2} d\delta_1 = \\ &= \sqrt{\sigma^2\tau}e^x \int_{-\infty}^{d_1} e^{-\frac{1}{2}\delta_1^2} d\delta_1 = \\ &= \sqrt{2\pi\sigma^2\tau}e^x\Phi(d_1) \end{aligned}$$

where $d_1 = \delta_1(0)$ as in equation (8.24) for $\tau = T - t$, and also, writing that $d_2 = \delta_2(0)$, as in equation (8.25) for $\tau = T - t$,

$$\begin{aligned} I_2 &= e^{-r\tau} \int_0^{\infty} e^{-\frac{1}{2}(\delta_2(s))^2} ds = \\ &= -\sigma\sqrt{\tau}e^{-r\tau} \int_{d_2}^{-\infty} e^{-\frac{1}{2}\delta_2^2} d\delta_2 =, \\ &= \sqrt{\sigma^2\tau}e^{-r\tau} \int_{-\infty}^{d_2} e^{-\frac{1}{2}\delta_2^2} d\delta_2 = \\ &= \sqrt{2\pi\sigma^2\tau}e^{-r\tau}\Phi(d_2) \end{aligned}$$

Thus

$$u(\tau, x) = K\left(e^x\Phi(d_1) - e^{-r\tau}\Phi(d_2)\right)$$

and transforming back $V(t,S) = u\left(T - t, \ln\left(S/K\right)\right)$ we get equation (8.23). □

Observe this is a backward parabolic PDE, which implies that the terminal condition determines the particular solution.

[8] We use integration by transformation of variables: if we define $z = \varphi(s)$ where $\varphi : [a,b] \to \mathcal{I}$ and $f : \mathcal{I} \to \mathbb{R}$ we have that

$$\int_{\varphi(a)}^{\varphi(b)} f(z)dz = \int_a^b f\left(\varphi(s)\right)\varphi'(s)ds.$$

8.7 Bibiography

- Mathematics of PDE's: introductory Olver (2014), Salsa (2016) and (Pinsky, 2003, ch 5). Advanced (Evans, 2010, ch 3).
- Applications to economics (with more advanced material) : Achdou et al. (2014)
- Applications to growth theory Brito (2004) and Brito (2011) and the references therein.

8.A Appendix: Fourier transforms

Consider a function $f(x)$ such that $x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} |f(x)| dx < \infty$. We can define a pair of generalized functions, the Fourier transform of $f(x)$, $F(s) = \mathcal{F}[f(x)](s)$ and the inverse Fourier transform $\mathcal{F}^{-1}[F(s)]\ (x) = f(x)$ (using the definition of Kammler (2000)), where

$$F(s) = \mathcal{F}[f(x)] \equiv \int_{-\infty}^{\infty} f(x)\, e^{-2\,\pi\, i\, s\, x}\, dx$$

where $i^2 = -1$ and

$$f(x) = \mathcal{F}^{-1}[F(s)] \equiv \int_{-\infty}^{\infty} F(s) e^{2\pi i s x} ds.$$

There are some useful properties of the Fourier transform that we use in the main text:

1. the Fourier transform preserves multiplication by a complex number $a \in \mathbb{C}$:

$$\mathcal{F}[a\, f(x)] \ = a\, F(s), \text{ and } \mathcal{F}^{-1}[aF(s)] \ = af(x),$$

Proof: $\mathcal{F}[a\, f(x)] \ = \int_{-\infty}^{\infty} a\, f(x)\, e^{-2\,\pi\, i\, s\, x}\, dx = a\, \int_{-\infty}^{\infty} f(x)\, e^{-2\,\pi\, i\, s\, x}\, dx = a\, F(s)$, and $\mathcal{F}^{-1}[a\, F(s)] \equiv \int_{-\infty}^{\infty} a\, F(s) e^{2\pi i s x} ds = a\, \int_{-\infty}^{\infty} F(s) e^{2\pi i s x} ds = a\, f(x)$;

2. the Fourier transform preserves linearity:

$$\mathcal{F}[a\, f(x) + b\, g(x)] \ = a\, F(s) + b\, G(s), \text{ and } \mathcal{F}^{-1}[a\, F(s) + b\, G(s)] \ = a\, f(x) + b\, g(x)$$

3. the Fourier transform does not preserve multiplication of two functions. However, there is a relationship between convolution of functions and multiplication of Fourier transforms. A **convolution** between two functions $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy.$$

The inverse Fourier transform of a product of two Fourier transforms is a convolution,

$$f(x) * g(x) = \mathcal{F}^{-1}[F(s)\, G(s)] \ = \int_{-\infty}^{\infty} F(s)\, G(s)\, e^{2\pi i s x} ds$$

4. $\mathcal{F}[x] = -\dfrac{1}{2\,\pi\, i}\delta'(s)$, where $\delta(x)$ is Dirac's delta. To prove this observe that

$$\int_{-\infty}^{\infty} e^{2\pi i s x} \delta(s) ds = 1$$

Therefore

$$
\begin{aligned}
x &= x \int_{-\infty}^{\infty} e^{2\pi i s x} \delta(s) ds \\
&= -\frac{1}{2\,\pi\,i} \left(\int_{-\infty}^{\infty} e^{2\,\pi\,i\,s\,x}\, \delta(s) - \int_{-\infty}^{\infty} 2\,\pi\,i\,x\, e^{2\,\pi\,i\,s\,x} \delta(s)\, ds \right) \\
&= -\frac{1}{2\,\pi\,i} \int_{-\infty}^{\infty} e^{2\,\pi\,i\,s\,x} \delta'(s)\, ds \\
&= -\frac{1}{2\,\pi\,i} \mathcal{F}^{-1}\Big[\ \frac{1}{2\,\pi\,i}\, \delta'(s)\Big]
\end{aligned}
$$

5. $\mathcal{F}[x^2] = \dfrac{1}{(2\,\pi)^2} \delta''(s)$

6. $\mathcal{F}[x\, f(x)] \ = -\dfrac{1}{2\,\pi\,i}\, F'(s)$
   Proof:

$$
\begin{aligned}
\mathcal{F}[x\, f(x)] \ &= \int_{-\infty}^{\infty} x\, f(x)\, e^{-2\,\pi\,i\,s\,x}\, dx \\
&= -\frac{1}{2\,\pi\,i} \ \int_{-\infty}^{\infty} -2\pi\, i\, x\, f(x)\, e^{-2\,\pi\,i\,s\,x}\, dx \\
&= -\frac{1}{2\,\pi\,i} \ \int_{-\infty}^{\infty} f(x)\, \frac{d}{ds}\Big(e^{-2\,\pi\,i\,s\,x} \Big)\, dx \\
&= -\frac{1}{2\,\pi\,i}\ \frac{d}{ds} F(s) = \frac{d}{ds} \left[\ \int_{-\infty}^{\infty} f(x)\, e^{-2\,\pi\,i\,s\,x}\, dx \right] \\
&= -\frac{1}{2\,\pi\,i}\, F'(s)
\end{aligned}
$$

   Alternative proof:

$$
\mathcal{F}[x\, f(x)] \ = \mathcal{F}[x] * \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} -\frac{1}{2\,\pi\,i}\, \delta'(y)\, F(s-y) dy = -\frac{1}{2\,\pi\,i}\, F'(s)
$$

7. if $f = f(x,t)$ where t is a real variable then $F(s,t) = \mathcal{F}[f(x,t)]$ and $f(x,t) = \ \mathcal{F}^{-1}[F(s,t)]$. Also $F_t(s,t) = \mathcal{F}[f_t(x,t)]$ and $f_t(x,t) = \ \mathcal{F}^{-1}[F_t(s,t)]$

8. $\mathcal{F}[f'(x)] \ = 2\,\pi\, i\, s\, F(s)$

Proof:

$$\begin{aligned}
\mathcal{F}[f'(x)] &= \int_{-\infty}^{\infty} f'(x)\, e^{-2\,\pi\, i\, s\, x}\, dx \\
&\text{integration by parts} \\
&= \int_{-\infty}^{\infty} f(x)\, e^{-2\,\pi\, i\, s\, x} - \int_{-\infty}^{\infty} f(x)\, \frac{\partial}{\partial x}\Big(e^{-2\,\pi\, i\, s\, x}\Big)\, dx \\
&\text{because } e^{-2\,\pi\, i\, s\, x} \text{ is symmetric the first integral is equal to zero} \\
&= 2\,\pi\, i\, s \int_{-\infty}^{\infty} f(x)\, e^{-2\,\pi\, i\, s\, x}\, dx \\
&= 2\,\pi\, i\, s\, F(s)
\end{aligned}$$

9. $\mathcal{F}[xf'(x)] = -\big(F(s) + sF'(s)\big)$ if $s \in \mathbb{R}$

   Proof:

$$\begin{aligned}
\mathcal{F}[x\, f'(x)] &= \mathcal{F}[x] * \mathcal{F}[f'(x)] \\
&= \int_{-\infty}^{\infty} \Big(-\frac{1}{2\,\pi\, i}\, \delta'(y)\big)\, \Big(2\,\pi\, i\, (s-y)\, F(s-y)\Big) dy \\
&= -\int_{-\infty}^{\infty} \delta'(y)\, (s-y)\, F(s-y) dy \\
&= -s \int_{-\infty}^{\infty} \delta'(y)\, F(s-y) dy + \int_{-\infty}^{\infty} \delta'(y)\, y\, F(s-y) dy \\
&= -sF'(s) + \int_{-\infty}^{\infty} \delta(y)\, y\, F(s-y) - \int_{-\infty}^{\infty} \delta(y)\, F(s-y) dy \\
&= sF'(s) - F(s)
\end{aligned}$$

10. $\mathcal{F}[f''(x)] = -4\,\pi^2\, s^2\, F(s)$
11. $\mathcal{F}[x\, f''(x)] = \dfrac{2\,\pi\, s}{i}\Big(2\, F(s) + s\, F'(s)\Big)$
12. $\mathcal{F}[x^2\, f''(x)] = -s^2\, F''(s)$

Some useful results:

Table 8.1: Fourier and inverse Fourier transforms of some functions

$f(x)$ for $-\infty < x < \infty$	$F(s)$ for $-\infty < s < \infty$	obs
$k\delta(x)$	k	k constant
k	$k\,\delta(s)$	k constant
$\delta(x-a)$	$e^{-2\,\pi\,i\,s\,a}$	
$\frac{1}{\sqrt{4\pi a}}e^{-\frac{x^2}{4a}}$	$e^{-a(2\pi s)^2}$	$a > 0$
$\frac{1}{\sqrt{4\pi a}}e^{-\frac{(x+b)^2}{4a}}$	$e^{-a(2\pi s)^2+b(2\pi i s)}$	$a > 0, \quad b \in \mathbb{R}$
$\frac{1}{\sqrt{4\pi a}}e^{c-\frac{x^2}{4a}}$	$e^{-a(2\pi s)^2+c}$	$a > 0, \quad c \in \mathbb{R}$
$\frac{1}{\sqrt{4\pi a}}e^{c-\frac{(x+b)^2}{4a}}$	$e^{-a(2\pi s)^2+b(2\pi i s)+c}$	$a > 0, \quad (b, c) \in \mathbb{R}^2$
$f(x) * g(x)$	$F(s)\,G(s)$	

Chapter 9

Optimal control of parabolic partial differential equations

9.1 Introduction

The optimal control of parabolic PDE is sometimes called optimal control of distributions.

Similarly to the optimal control of ODE's, the first order conditions include a system of forward-backward parabolic PDE's together with boundary conditions. The requirements for the existence of solutions are clearly very strong, because the general solutions of the PDE system may not allow for the boundary conditions to be satisfied. Ill-posedness is, therefore, an important issue here.

Next we present the necessary conditions for three different optimal control of parabolic PDE's: a simple infinite horizon problem in section 9.2, an average optimal control problem in subsection 9.2.1, and the optimal control of a Fokker-Planck-Kolmogorov equation in section 9.3.

9.2 A simple optimal control problem

Next we consider a simple optimal control problem for a system governed by a parabolic PDE.

We have two independent variables, time $t \in \mathbb{R}_+$ and another independent variable $x \in \mathbb{R}$ and two dependent functions, the control $u = u(t, x)$, mapping $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and a state $y = y(t, x)$, mapping $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$.

The system to be controlled is given by a semi-linear parabolic partial differential equation $y_t = y_{xx} + g(t, x, u, y)$ where $g(\cdot, u, y)$ is smooth, by an initial condition $y(0, x) = y_0(x)$, where function $y_0 : \mathbb{R} \to \mathbb{R}$ is and bounded, and a Neumann boundary condition $\lim_{x\to\pm\infty} y_x(t, x) = 0$ is given for every t. The boundary condition means that the state variable should be "flat" for very large absolute values of variable x.

The utility functional involves both integration in time and in the other independent variable

$$J[u,y] \ = \int_0^\infty \int_{-\infty}^\infty f(t,x,u(t,x),y(t,x))\,dx\,dt$$

where we assume that $f(\cdot,u,y)$ is smooth and measurable, in the sense [1]

$$\int_{\mathbb{R}_+\times\mathbb{R}} |f(t,x)|d(t,x) < \infty.$$

Therefore our problem is to find the optimal $u^* = \big(u^*(t,x)\big)_{(t,x)\in\mathbb{R}_+\times\mathbb{R}}$ and $y^* = \big(y^*(t,x)\big)_{(t,x)\in\mathbb{R}_+\times\mathbb{R}}$ that solve the problem:

$$\max_{u(\cdot)} \int_0^\infty \int_{-\infty}^\infty f(t,x,u(t,x),y(t,x))dxdt \tag{9.1}$$

subject to the constraints

$$\begin{cases} \dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2} + g(t,x,u(t,x),y(t,x)), & \text{for } (t,x)\in\mathbb{R}_+\times\mathbb{R} \\ y(0,x) = y_0(x), & \text{for } (t,x)\in\{\ t=0\}\ \times\mathbb{R} \\ \lim_{x\to\pm\infty} \dfrac{\partial y(t,x)}{\partial x} = 0 & \text{for } (t,x)\in\mathbb{R}_+\times\{(x=-\infty),(x=\infty)\ \} \end{cases} \tag{9.2}$$

Next we find the optimality conditions for this problem applying a distributional Pontriyagin maximum principle.

We define the Hamiltonian

$$H(t,x,u,y,\lambda) = f(t,x,u,y) + \lambda(t,x)g(t,x,u,y)$$

and call $\lambda = \lambda(t,x)$ the co-state variable.

Proposition 1 (**Necessary first-order conditions**)**.** *Let* (u^*,y^*) *be a solution to problem* (9.1)-(9.2)*. Then there is a co-state variable* λ *such that the following necessary conditions hold*

$$\frac{\partial H^*(t,x)}{\partial u} \ = \ 0 \tag{9.3}$$

$$\frac{\partial \lambda(t,x)}{\partial t} \ = \ -\frac{\partial^2\lambda(t,x)}{\partial x^2} - \frac{\partial H^*(t,x)}{\partial y} \tag{9.4}$$

$$\lim_{t\to\infty} \lambda(t,x) \ = \ 0 \tag{9.5}$$

$$\lim_{x\to\pm\infty} \frac{\partial \lambda(t,x)}{\partial x} \ = \ 0. \tag{9.6}$$

together with equations (9.2)

[1] This condition requires that the function is bounded for every value of $u(.)$ and $y(.)$ and allow for the use of Fubini's theorem, i.e, for the interchange of the integration for t and x. Intuitively, we should consider functions such that the order of integration does not matter.

See the proof in the appendix. Equation (9.3) is a static optimality condition, that if function $H(u, .)$ is sufficiently smooth, allows for the determination of the optimal control u^* as a function of the co-state variable, and the state variable. Equation (9.4) is a Euler-equation. In this case it is a backward parabolic PDE which encodes the incentives for changing the control variable. Equations (9.5) and (9.6) are transversality conditions, which are dual to the boundary conditions in (9.2) related to the asymptotic properties of the solution.

If functions $f(\cdot)$ and $g(\cdot)$ are sufficiently smooth in (u, y) , we can use the implicit function theorem to obtain from equation (9.3)

$$u^* = U(t, x, y(t, x), \lambda(t, x)),$$

yielding

$$G(t, x, y(t, x), \lambda(t, x)) = g(t, x, u^*(t, x), y(t, x))$$

and

$$L(t, x, y(t, x), \lambda(t, x)) = f_y(t, x, u^*(t, x), y(t, x)) + \lambda(t, x)\, g_y(t, x, u^*(t, x), y(t, x))$$

we have a distributional MHDS system

$$\begin{cases} \dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2} + G(t, x, y, \lambda), & \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\ \dfrac{\partial \lambda}{\partial t} = -\dfrac{\partial^2 \lambda}{\partial x^2} - L(t, x, y, \lambda) & \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \end{cases}$$

This system has two semi-linear parabolic PDE's: a forward parabolic PDE for the state variable and a backward parabolic PDE for the co-state variable. It is a distributional generalization of the MHDS for an optimal control problem of ODE's.

9.2.1 Average optimal control problem

Next we consider a particular case of the previous problem in which the planner maximizes the present-value of an average utility function

$$J(y, u) = \lim_{x \to \infty} \frac{1}{2x} \int_{-x}^{x} \int_0^{\infty} f(y(\xi, t), u(\xi, t)) e^{-\rho t} dt d\xi \tag{9.7}$$

where $\rho > 0$ and $f(.)$ is continuous and differentiable. We assume the same semi-linear parabolic constraint.

The problem is

$$\max_u \lim_{x \to \infty} \frac{1}{2x} \int_{-x}^{x} \int_0^{\infty} f(y(\xi, t), u(\xi, t)) e^{-\rho t} dt d\xi$$

subject to

$$y_t = \sigma^2 y_{xx} + g(y, u),\ (t, x) \in \mathbb{R}_+ \times \mathbb{R} \tag{9.8}$$

$$y(x, 0) = \phi(x),\ x \in \mathbb{R} \tag{9.9}$$

$$\lim_{t \to \infty} R(t) y(t, x) \geq 0,\ \ x \in \mathbb{R} \tag{9.10}$$

$$\lim_{x \to \pm\infty} \frac{y(t, x)}{x} = 0,\ t \in \mathbb{R}_+. \tag{9.11}$$

where $R(t) \le R_0 e^{-\rho t}$, where h_0 is a constant.

The (current-value) Hamiltonian function is

$$H(y,u,q) \equiv f(y,u) + qg(y,u)$$

where $q(t,x)$ is the current value co-state variable.

The necessary first order conditions, according to the Pontryagin's maximum principle are the following.

Proposition 2 (Necessary conditions for the optimal average problem)**.** *Assume there are optimal processes for the state and the control variable, $y^* = (y^*(t,x))_{(t,x)\in\mathbb{R}\times\mathbb{R}_+}$ and $u^* = (u^*(t,x))_{(t,x)\in\mathbb{R}\times\mathbb{R}_+}$ then there is a (current-value) co-state variable $q(t,x)$ such that the following conditions hold:*

- *the optimality condition*

$$\frac{\partial H}{\partial u}\ (y^*(t,x), u^*(t,x), q(t,x)\ ) = 0,\ (t,x) \in (t,x) \in \mathbb{R}_{++} \times \mathbb{R}$$

- *the distributional Euler equation*

$$q_t = -\sigma^2 q_{xx} + q\,(\rho - \frac{\partial H}{\partial y}\ (y^*(t,x), u^*(t,x), q(t,x)\ ))),\ (t,x) \in (t,x) \in \mathbb{R}_{++} \times \mathbb{R}$$

- *the boundary condition, dual to equation* (9.11)

$$\lim_{x\to\pm\infty} e^{-\rho t}\frac{q(t,x)}{x} = 0,\ t \in \mathbb{R}_{++}$$

- *the transversality condition*

$$\lim_{t\to\infty} e^{-\rho t} \lim_{x\to\infty} \int_{-x}^{x} q(\xi,t)y(\xi,t) = 0,\ \{t = \infty$$

See the proof in the Appendix

9.2.2 Application: the distributional AK model

As application consider a simple model in which there is a central planner in a dynastic economy who wants to maximize the average (un-weighted) utility of an economy composed with heterogeneous agents, distributed in space from a central point $x = 0$. Assume that the heterogeneity is only given by their initial asset position, $k_0(x)$. Each agent produces a different quantity of a good, depending only on their endowment of capital and the central planner assigns consumption which varies between consumers and can be different from their production (given the capital endowment). Therefore there is a distribution of savings in the economy allowing some agents to use more (less) capital than they have at the beginning of every (infinitesimal) period. This section draws upon Brito (2004) and Brito (2011), which present this model with more detail.

Consider agents located at r and having the capital stock $K(t,r)$ and having savings $S(t,r)$ at time t. Savings is equal to income minus consumption, where we assume that income is generated by a linear production function

$$Y(t,r) = A\,K(t,r).$$

Savings can be applied in the own region, $I(t,r)$, or in other regions $T(t,r)$: therefore $S(t,r) = I(t,r) + T(t,r)$ where is trade balance. If there is no depreciation then $I(t,s) = \frac{\partial K}{\partial t}$. We order the regions according to their capital endowment then x can be used as a index for the regions. If, in addition, we consider that: first, the flow of capital runs from regions with high capital intensity to regions to low capital intensity and, second, that the flow is proportional to the gradient of the capital intensity at the boundary of region $r = [x, x + \Delta x]$, then flow

$$T(t,r) = \tau^2 \int_x^{\Delta x} \frac{\partial K}{\partial x}(t,s)\,ds.$$

If we let $\Delta x \to 0$ then we find distributional capital accumulation constraint for every location x

$$\frac{\partial K(t,x)}{\partial t} = \tau^2 \frac{\partial^2 K(t,x)}{\partial x^2} + AK(t,x) - C(t,x)\ \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{9.12}$$

The problem is to maximize the average intertemporal discounted utility of consumption, C,

$$J(C,K) \equiv \max_{[C]} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_0^{\infty} \frac{C(\xi,t)^{1-\theta}}{1-\theta}\, e^{-\rho t} dt d\xi. \tag{9.13}$$

subject to the distributional capital accumulation equation (9.12) and the terminal, the boundary and the initial conditions

$$\lim_{t\to\infty} e^{-\int_0^t r(x,s)ds} K(t,x) \geq 0, \quad \forall x \in \mathbb{R} \tag{9.14}$$

$$\lim_{x\to\mp\infty} \frac{K(t,x)}{x} = 0, \quad \forall t \in \mathbb{R}_+ \tag{9.15}$$

$$K(x,0) = \phi(x), \ \forall x \in \mathbb{R} \ \text{given.} \tag{9.16}$$

According to the distributional Pontriyagin maximum principle (see the Appendix) the distributional MHDS is

$$\frac{\partial K}{\partial t} = \tau^2 \frac{\partial^2 K}{\partial x^2} + AK - C, \ \ x \in \mathbb{R},\ t > 0 \tag{9.17}$$

$$\frac{\partial C}{\partial t} = -\tau^2 \left[\frac{\partial^2 C}{\partial x^2} - \frac{1+\theta}{C} \left(\frac{\partial C}{\partial x} \right)^2 \right] + \gamma C, \ \ x \in \mathbb{R},\ t > 0 \tag{9.18}$$

where the endogenous growth rate i s

$$\gamma \equiv \frac{A-\rho}{\theta}, \tag{9.19}$$

the transversality condition

$$\lim_{t\to\infty}\lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x} e^{-\rho t}K(\xi,t)C(\xi,t)^{-\theta}d\xi = 0, \tag{9.20}$$

and the dual boundary conditions

$$\lim_{x\to\pm\infty}\left(e^{\rho t}C(t,x)^{\theta}x\right)^{-1} = 0,\ t\geq 0. \tag{9.21}$$

In system (9.18)-(9.17), A is the net total factor productivity and γ is equal to the endogenous growth rate in the benchmark homogeneous AK model. The initial condition $K(x,0)=\phi(x)$ and the boundary condition (9.15) should also hold.

The coupled system (9.18)-(9.17) has the closed form solution

$$K(t,x) = e^{\gamma t}k(t,x),\ C(t,x) = e^{\gamma t}c(t,x),\ t\geq 0,\ x\in\mathbb{R} \tag{9.22}$$

where

$$k(t,x) = \frac{1}{2\tau\sqrt{\pi\theta t}}\int_{-\infty}^{\infty}\phi(\xi)e^{-\left(\frac{x-\xi}{2\tau}\right)^2\frac{1}{\theta t}}d\xi,\quad t>0,\ x\in\mathbb{R}$$

and

$$c(t,x) = \frac{1}{2\tau\sqrt{\pi\theta t}}\int_{-\infty}^{\infty}\phi(\xi)\left[r-\gamma+(\theta-1)\left(\frac{1}{2\theta t}-\left(\frac{x-\xi}{2\tau\theta t}\right)^2\right)\right]\ e^{-\left(\frac{x-\xi}{2\tau}\right)^2\frac{1}{\theta t}}\,d\xi,\quad t>0,\ x\in\mathbb{R}.$$

A necessary condition for the existence of a solution of the centralized problem is that $A>\gamma$. This model displays convergence to a time-unbounded balanced growth path similar to the homogeneous-agent AK model: capital will be equalized among regions. Figure **??** presents a graphic depiction of the detrended-solution for $\phi(x)=e^{-|x|}$. We observe that the initial heterogeneity is eliminated along the transition

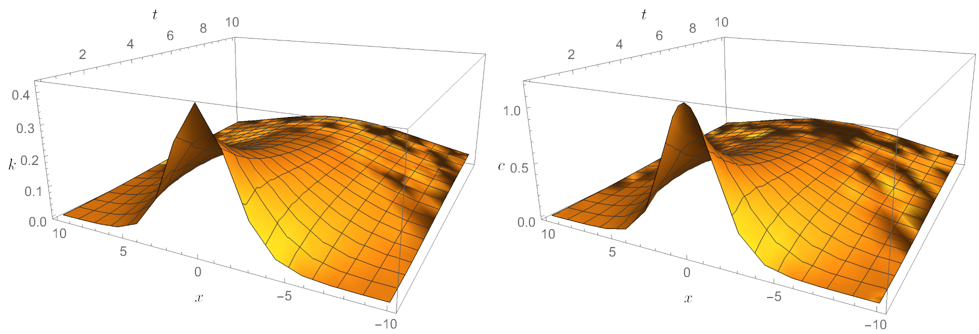

Figure 9.1: Convergence to a homogeneous asymptotic: local dynamics for the detrended $k(t,x)$ and $c(t,x)$ for the case AK.

9.3 Optimal control of the Fokker-Planck-Kolmogorov equation

In this section we consider a problem in which the PDE constraint of the economy is represented by a Fokker-Planck-Kolmogorov equation, which, as we saw models probabilities of distributions across time.

This problem has two particularities: first, the transport and the diffusion terms are controled endogeneously by the. control variable, second, the state variable enters in the objective functional as a weighting variable.

In principle, this problem has other particular properties that should be highlighted:

1. the boundary conditions (9.25) introduce an initial condition is a density function such that

$$\int_X y_0(x)\, dx = 1$$

   and for every point in time the density zero in the extremes of the support. This implies that the a conservation law should hold for every $t \in T$,

$$\int_X y(t,x)\, dx = 1$$

   and that the state variable is bounded in X for every point in time (it is a L^2 function);

However, in order to have this conservative property several technical problems have to be solved. Although first-order PDE's satisfy a conservation law for $t > 0$ if the initial condition, for $t = 0$ does satisfy it, this property does not hold generally for parabolic PDE's. Some normalization has to be introduced in the solution for single equations. We are not aware of the effect of this on the solution to optimal control problem.

Therefore, the version we present next does not necessary satisfy a conservation law.

Let $T = [\underline{t}, \overline{t}]$ and $X = [\underline{x}, \overline{x}]$, the state variable $y : T \times X \to \mathbb{R}$ and the control variable $u : T \times X \to \mathbb{R}$.

We consider the optimal control problem of a Fokker-Planck equation

$$\max_{u(\cdot)} \int_T \int_X f(t,x,u(t,x))\, y(t,x)\, dx\, dt \tag{9.23}$$

subject to

$$\partial_t y(t,x) + \partial_x \Big(g(x,u(t,x))\, y(t,x) \Big) - \partial_{xx} \Big(h(x,u(t,x))\, y(t,x) \Big) = 0, \text{ a.e. } (t,x) \in T \times X \tag{9.24}$$

and the boundary constraints

$$\begin{cases} y(\underline{t},x) = y_0(x), \ (t,x) \in \{t = \underline{t}\} \ \times X \\ y(t,x) = 0, \ (t,x) \in T \ \times \{x = \underline{x}, x = \overline{x}\} \end{cases} \tag{9.25}$$

We assume that functions $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are continuous and continuously differentiable as regards the control variable $u(\cdot)$.

Proposition 3 (Optimal control of the FPK equation).
Let $u^*(\cdot)$ *and* $y^*(\cdot)$ *be the solution of problem* (9.23)-(9.24)-(9.25). *Then there is a function* $\lambda : T \times X \to \mathbb{R}$ *such that:*

1. *the optimality condition*

$$\partial_u g(x, u^*(t,x))\partial_x \lambda(t,x) + \partial_u h(x, u^*(t,x))\partial_{xx}\lambda(t,x) + \partial_u f(t, x, u^*(t,x)) = 0 \ a.e\ (t,x) \in T \times X \tag{9.26}$$

2. *the distributional Euler equation*

$$\partial_t \lambda(t,x) + g(x, u^*(t,x))\partial_x \lambda(t,x) + h(x, u^*(t,x))\partial_{xx}\lambda(t,x) + f(t, x, u^*(t,x)) = 0 \ a.e\ (t,x) \in T \times X \tag{9.27}$$

3. *the transversality condition*

$$\lambda(\bar{t}, x) = 0, \ (t,x) \in \{t = \bar{t}\} \ \times X$$

4. *and the admissibility constraints* (9.24) *and* (9.25) *evaluated at the optimum.*

9.3.1 Application: optimal distribution of capital with stochastic redistribution

Consider an economy with heterogeneous households and that the household with capital stock $k(t)$, at time t has the accumulation equation

$$dk(t) = (Ak(t) - c(t))\, dt + \sigma k(t) dW(t)$$

where dW is a Wiener process. We assume that $k \in [0, \infty)$. If $n(t,k)$ is the density of households with capital k at time t the distribution function for households satisfies

$$\int_0^\infty n(t,k)dt = 1, \text{for every}\, t \in [0, \infty).$$

Therefore, the distribution of households satisfies the FPK equation

$$\partial_n(t,k) + \partial_k \Big((Ak - c)\, n(t,k) \Big) - \frac{1}{2}\partial_{kk} \ \Big((\sigma k)^2\, n(t,k) \Big) = 0$$

where $n(0,k) = \phi(k)$ is given and $n(t,0) = \lim_{k\to\infty} n(t,k) = 0$.

We assume a central planer wants to allocate consuming among households, and through time, in order to maximize a social welfare function. We assume the social welfare function is

$$\int_0^\infty \int_0^\infty \ln\left(c(k,t)\right) n(k,t) e^{-\rho t}\, dk\, dt, \ \rho > 0$$

Applying Proposition 3 the necessary first order conditions lead to the forward-backward parabolic PDE-system

$$\partial_t q(t,k) + Ak\,\partial_k\, q(t,k) + \ln\left((\partial_k q(t,k))^{-1}\right) + \frac{1}{2}\,\sigma^2 k^2 \partial_{kk} q(t,k) - \rho q(t,k) - 1 = 0 \quad (9.28)$$

$$\partial_t n(t,k) + \left(\left(Ak - (\partial_k q(t,k))^{-1}\right) n(t,k)\right)_k - \frac{\sigma^2}{2}\partial_{kk}\left(k^2 n(t,k)\right) = 0 \quad (9.29)$$

together with a transversality condition

$$\lim_{t\to\infty} e^{-\rho t} q(t,k) = 0.$$

where $q(t,k) = e^{\rho t}\lambda(t,k)$ is the current distributional co-state variable.

Because the system is recursive, we can solve the Euler equation together with the transversality condition for $q(t,k)$. Substituting in the constraint (9.29) we obtain a linear parabolic PDE for the optimal dynamics of the distribution

$$\partial_t n(k,t) + \partial_k\left(\gamma k n(k,t)\right) - \frac{\sigma^2}{2}\partial_{kk}\left(k^2 n(k,t)\right) = 0.$$

Solving the Cauchy problem with $n(0,k) = \phi(k)$ a closed form solution can be obtained

$$n^*(t,x) = \int_0^\infty \phi(\xi)\, g\left(t, \ln\left(\frac{k}{\xi}\right)\right)\frac{1}{\xi} d\xi. \quad (9.30)$$

where

$$g(t,y) = (2\pi\sigma^2 t)^{-\frac{1}{2}} \exp\left[(\gamma - \sigma^2)t - \frac{\left(y - t(\gamma - \frac{3}{2}\,\sigma^2)\right)^2}{2\sigma^2 t}\right],\ x \in \mathbb{R}.\ . \quad (9.31)$$

This solution contains both a transport mechanism, which tends to generate growth at a rate $\gamma \equiv A - \rho > 0$ and a diffusion mechanism, with strength σ^2. We can show that the average capital stock is

$$M_k(t) = \int_0^\infty k\, n(t,k)\, dk = M_k(0)\, e^{(\gamma-\sigma^2)t},\ t \in [0,\infty)$$

meaning that there is long run growth if $\gamma - \sigma^2 > 0$, that is if the stochastic distribution of growth is not too volatile.

9.4 References

- Optimal control problem of partial differential equations or an optimal distributed control problem Butkovskiy (1969), Lions (1971), Derzko et al. (1984) or Neittaanmaki and Tiba (1994) present optimality results with varying generality. We draw mainly upon the last two references. See also the textbooks: Fattorini (1999) and Tröltzsch (2010).

- Applications in economics Carlson et al. (1996, chap.9)

9.A Proofs

Next we present heuristic proofs of the three versions of the distributional PMP presented in the main text.

Proof of Proposition 1. The value functional is

$$V[u,y] = \int_0^\infty \int_{-\infty}^\infty f(t,x,u(t,x),y(t,x))dxdt,$$

considering the constraint we have

$$\begin{aligned}
V[u,y] &= \int_0^\infty \int_{-\infty}^\infty \left[f(t,x,u(t,x),y(t,x)) + \lambda(t,x)\left(g(t,x,u(t,x),y(t,x)) - \frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial x^2}\right)\right] dx\, dt = \\
&= \int_0^\infty \int_{-\infty}^\infty \left[H(t,x,u(t,x),y(t,x),\lambda(t,x)) - \lambda(t,x)\frac{\partial y}{\partial t} + \lambda(t,x)\frac{\partial^2 y}{\partial x^2}\right] dx\, dt \\
&= \int_0^\infty \int_{-\infty}^\infty \left[H(t,x,u(t,x),y(t,x),\lambda(t,x)) + \left(\frac{\partial \lambda(t,x)}{\partial t} + \frac{\partial^2 \lambda(t,x)}{\partial x^2}(t,x)\right) y(t,x)\right] dx\, dt + \\
&- \int_{-\infty}^\infty \lambda(t,x)y(t,x)\, dx\Big|_{t=0}^\infty + \int_0^\infty \left(\lambda(t,x)\frac{\partial y(t,x)}{\partial x} - \frac{\partial \lambda(t,x)}{\partial x}y(t,x)\right) dt\Big|_{x=-\infty}^\infty
\end{aligned}$$

for any control and state variables.

Let us assume we know the optimal control and state variables $u^*(t,x)$ and $y^*(t,x)$ then

$$V[u^*,y^*] = \int_0^\infty \int_{-\infty}^\infty f(t,x,u^*(t,x),y^*(t,x))dxdt,$$

and let $u(t,x)$ and $y(t,x)$ be admissible perturbations over the optimal levels

$$\begin{aligned}
u(t,x) &= u^*(t,x) + \varepsilon h_u(t,x) \\
y(t,x) &= y^*(t,x) + \varepsilon h_y(t,x)
\end{aligned}$$

where ϵ is a constant, and $h_y(0,x) = 0$, for every $x \in \mathbb{R}$, and $\lim_{x\pm\infty} \dfrac{\partial h_y(t,x)}{\partial x} = 0$, for every $t \in \mathbb{R}_+$.

The integral derivative evaluated at $\varepsilon = 0$ is

$$\begin{aligned}
\delta V[u^*,y^*] &= \int_0^\infty \int_{-\infty}^\infty \left[\frac{\partial H^*(t,x)}{\partial u} h_u(t,x) + \left(\frac{\partial H^*(t,x)}{\partial y} + \frac{\partial \lambda(t,x)}{\partial t} + \frac{\partial^2 \lambda(t,x)}{\partial x^2}\right) h_y(t,x)\right] dx\, dt - \\
&- \int_{-\infty}^\infty \lambda(t,x)h_y(t,x)\, dx\Big|_{t=0}^\infty + \int_0^\infty \lambda(t,x)\frac{\partial h_y(t,x)}{\partial x} - \frac{\partial \lambda(t,x)}{\partial x}h_y(t,x)\Big)\, dt \quad {}_{x=-\infty}^{\infty} =
\end{aligned}$$

where $H^*(t,x) = H(t,xu^*(t,x),y^*(t,x),\lambda(t,x))$. From admissibility conditions, we have

$$\begin{aligned}
\delta V[u^*,y^*] &= \int_0^\infty \int_{-\infty}^\infty \left[\frac{\partial H^*(t,x)}{\partial u} h_u(t,x) + \left(\frac{\partial H^*(t,x)}{\partial y} + \frac{\partial \lambda(t,x)}{\partial t} + \frac{\partial^2 \lambda(t,x)}{\partial x^2}\right) h_y(t,x)\right] dxdt - \\
&- \lim_{t\to\infty} \int_{-\infty}^\infty \lambda(t,x)h_y(t,x)dx - \int_0^\infty \left(\frac{\partial \lambda(t,x)}{\partial x}h_y(t,x)\right) dt\Big|_{x=-\infty}^\infty
\end{aligned}$$

Optimality requires that $V[u^*, y^*] \geq V[u, y]$ which holds only if $\delta V[u^*, y^*] = 0$. Then optimality conditions are as in equation (9.2). $\square$

Proof of Proposition 2. Let us assume that there is a solution (u^*, y^*), for the problem, and define the value function as

$$V[u^*, y^*] = \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_0^{\infty} f(y^*(\xi, t), u^*(\xi, t)) e^{-\rho t} dt d\xi.$$

Consider a small continuous perturbation $(u(\epsilon), y(\epsilon)) = \{(u(t, x), y(t, x)) : (t, x) \in \mathbb{R} \times \mathbb{R}_+\}$, where ϵ is any positive constant, such that $u(t, x) = u^*(t, x) + \epsilon h_u(t, x)$ and $y(t, x) = y^*(t, x) + \epsilon h_y(t, x)$, for $t > 0$, and $h_u(x, 0) = h_y(x, 0) = 0$, for every $x \in \mathbb{R}$. The value of this strategy is

$$V(\epsilon) = \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_0^{\infty} f(u(\xi, t), y(\xi, t)) e^{-\rho t} dt d\xi.$$

But,

$$\begin{aligned}
V(\epsilon) := \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_0^{\infty} f(u(\xi, t), y(\xi, t)) e^{-\rho t} dt d\xi - \\
- \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_0^{\infty} \lambda(\xi, t) \left[\frac{\partial y(\xi, t)}{\partial t} - \frac{\partial^2 y(\xi, t)}{\partial \xi^2} - g(u(\xi, t), y(\xi, t)) \right] dt d\xi + \\
+ \lim_{t\to\infty} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} e^{-r(\xi, t)} \mu(\xi, t) y(\xi, t) d\xi \quad (9.32)
\end{aligned}$$

where $\lambda(.)$ is the co-state variable and $\mu(.)$ is a Lagrange multiplier associated with the solvability condition. In the optimum, the Kuhn-Tucker condition should hold

$$\lim_{t\to\infty} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} e^{-r(\xi, t)} \mu(\xi, t) y(\xi, t) d\xi = 0.$$

By using integration by parts we find that

$$\int_0^{\infty} \lambda(t, x) \frac{\partial y(t, x)}{\partial t} dt = \lambda(t, x) y(t, x) \Big|_{t=0}^{\infty} - \int_0^{\infty} \frac{\partial \lambda(t, x)}{\partial t} y(t, x) dt$$

and that

$$\begin{aligned}
\int_{-x}^{x} \int_0^{\infty} \lambda(\xi, t) \frac{\partial^2 y(\xi, t)}{\partial \xi^2} dt d\xi = \\
= \int_0^{\infty} \lambda(\xi, t) \frac{\partial y(\xi, t)}{\partial \xi} \Big|_{\xi=-x}^{x} - y(\xi, t) \frac{\partial \lambda(\xi, t)}{\partial \xi} \Big|_{\xi=-x}^{x} dt + \int_{-x}^{x} \int_0^{\infty} \frac{\partial^2 \lambda(\xi, t)}{\partial \xi^2} y(\xi, t) dt d\xi, \quad (9.33)
\end{aligned}$$

where the second term is canceled by the boundary conditions (9.15). Then

$$\begin{aligned}
V(\epsilon) &= \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_0^{\infty} \Big(f(u(\xi, t), y(\xi, t)) e^{-\rho t} + \\
&+ \frac{\partial \lambda(\xi, t)}{\partial t} y(\xi, t) + \frac{\partial^2 \lambda(\xi, t)}{\partial \xi^2} y(\xi, t) + \lambda(\xi, t) g(u(\xi, t), y(\xi, t)) \Big) dt d\xi - \\
&- \lim_{x\to\infty} \frac{1}{2x} \left(\int_{-x}^{x} \lambda(\xi, t) y(\xi, t) \Big|_{t=0}^{\infty} d\xi + \int_0^{\infty} \lambda(\xi, t) \frac{\partial y(\xi, t)}{\partial \xi} \Big|_{\xi=-x}^{x} dt \right)
\end{aligned}$$

If an optimal solution exists, then we may characterize it by applying the variational principle,

$$\frac{\partial J(u^*, y^*)}{\partial \epsilon} = \lim_{\epsilon \to 0} \frac{J(u(\epsilon), y(\epsilon)) - J(u^*, y^*)}{\epsilon} = 0.$$

But, defining the Hamiltonian function as $H(u, y, \lambda) = f(u, y) + \lambda g(u, y)$, then

$$\begin{aligned} \frac{\partial J}{\partial \epsilon} = \lim_{x \to \infty} \frac{1}{2x} \Bigg\{ \int_{-x}^{x} \int_{0}^{\infty} & [\ H_u(u^*(\xi,t), y^*(\xi,t), \lambda(\xi,t)) h_u(\xi,t) + \\ & + \left(\frac{\partial \lambda(\xi,t)}{\partial t} + \frac{\partial^2 \lambda(\xi,t)}{\partial \xi^2} + \lambda(\xi,t) H_x(u^*(\xi,t), y^*(\xi,t), \lambda(\xi,t)) \right) h_y(\xi,t) \Bigg] dt d\xi \\ & - \int_{-x}^{x} \lambda(\xi,t) h_y(\xi,t) \Big|_{t=0}^{\infty} d\xi + \int_{0}^{\infty} \lambda(\xi,t) \frac{\partial h_y(\xi,t)}{\partial \xi} \Big|_{\xi=-x}^{x} dt - \\ & - \lim_{t \to \infty} \int_{-x}^{x} \mu(\xi,t) e^{-r(\xi,t)} h_y(\xi,t) d\xi \Bigg\}. \end{aligned} \tag{9.34}$$

The last and the third to last expressions are canceled if $\lim_{t\to\infty}[\mu(t,x)e^{-r(t,x)} - \lambda(t,x)] = 0$, and by the fact that $h_k(x, 0) = 0$, for any x. Then, substituting in the Kuhn-Tucker condition we get a generalized transversality condition. We get the first order conditions by equating to zero all the remaining components of $\frac{\partial J}{\partial \epsilon}$. Equations (**??**)-(**??**) are obtained by simply making $q(t,x) = e^{\rho t}\lambda(t,x)$. □

Proof of Proposition 3. The Lagrange functional is

$$\begin{aligned} L[u, y, \lambda] &= \int_T \int_X f(t, x, u(t,x))\, y(t,x)\, dx\, dt \\ &- \int_T \int_X \lambda(t,x) \partial_t y(t,x)\, dx\, dt \\ &- \int_T \int_X \lambda(t,x) \partial_x \Big(g(x, u(t,x))\, y(t,x) \Big), dx\, dt \\ &+ \int_T \int_X \lambda(t,x) \partial_{xx} \Big(h(x, u(t,x))\, y(t,x) \Big), dx\, dt \\ &\equiv I_1 - I_2 - I_3 + I_4 \end{aligned}$$

Integrating by parts, we find

$$I_2 = \int_X \lambda(t,x)\, y(t,x) dx \Big|_{t \in \partial T} - \int_T \int_X \partial_t \lambda(t,x)\, y(t,x) dx\, dt,$$

where ∂T is the boundary of T, i.e, $\partial T = \{\ \underline{t}, \bar{t}\}$,

$$I_3 = \int_T \lambda(t,x)\, g(x, u(t,x))\, y(t,x) \Big)\, dt \Big|_{x \in \partial X} - \int_T \int_X g(x, u(t,x))\, \partial_x \lambda(t,x)\, y(t,x) dx\, dt$$

and

$$\begin{aligned} I_4 = & \int_T \left[\lambda(t,x)\, \partial_x \Big(h(x, u(t,x))\, y(t,x) \Big) - \partial_x \lambda(t,x)\, h(x, u(t,x))\, y(t,x) \right] dt \Big|_{x \in \partial X} \\ & + \int_T \int_X h(x, u(t,x))\, y(t,x)\, \partial_{xx} \lambda(t,x)\, dx\, dt. \end{aligned}$$

Therefore, the Lagrange functional becomes

$$
\begin{aligned}
L[u,y,\lambda] = \int_T \int_X \Bigg[& f(t,x,u(t,x)) + \partial_t \lambda(t,x) + g(x,u(t,x))\,\partial_x \lambda(t,x) + \\
& + h(x,u(t,x))\,\partial_{xx}\lambda(t,x) \Bigg] \; y(t,x)\,dx\,dt \\
& - \int_X \lambda(t,x)\,y(t,x)dx \Big|_{t\in\partial T} \\
& - \int_T \lambda(t,x)\,g(x,u(t,x))\,y(t,x)\,dt \Big|_{x\in\partial X} \\
& + \int_T \Bigg[\lambda(t,x)\,\partial_x\Big(h(x,u(t,x))\,y(t,x)\Big) - \partial_x\lambda(t,x)\,h(x,u(t,x))\,y(t,x) \Bigg] \; dt \Big|_{x\in\partial X} \\
& = I_5 - I_6 - I_7 + I_8
\end{aligned}
$$

The Lagrange functional at the optimal control and state variables, $u^*(\cdot)$ and $y^*(\cdot)$ is written $L^* = L[u^*, y^*, \lambda]$.

We Introduce perturbations on the control and state variables,

$$
\begin{aligned}
u(t,x) &= u^*(t,x) + \varepsilon\,\delta_u(t,x),\ (t,x) \in T \times X \\
y(t,x) &= y^*(t,x) + \varepsilon\,\delta_y(t,x),\ (t,x) \in T \times X
\end{aligned}
$$

which are admissible if $\delta_y(\underline{t},x) = 0$ for any $x \in X$, and $\delta_y(t,x) = 0$ for $x \in \partial X$ and $\delta_y(\bar{t},x)$ is arbitrary for $x \in \mathrm{Int}(X)$.

A necessary condition for the optimum is that the integral derivative (Gâteux derivative) of $L[u,y,\lambda]$, evaluated at $u^*(\cdot)$ and $y^*(\cdot)$, $L^* = L[u^*, y^*, \lambda]$ is zero $\delta L^* = 0$. As I_5, I_6, I_7 and I_8 are also functionals over $u(\cdot)$, $y(\cdot)$ and $\lambda(\cdot)$, this is equivalent to requiring the functional derivatives to be zero when evaluated at $u^*(\cdot)$ and $y^*(\cdot)$.

The first functional derivative is

$$
\begin{aligned}
\delta I_5^* = \int_T \int_X & \Bigg[\partial_u \ f(t,x,u^*(t,x)) + \partial_u g(x,u^*(t,x))\,\partial_x\lambda(t,x) + \partial_u h(x,u^*(t,x)\,\partial_{xx}\lambda(t,x) \Bigg] \; y^*(t,x)\,\delta_u(t,x) \\
& + \Bigg[f(t,x,u^*(t,x)) + \partial_t\lambda(t,x) + g(x,u^*(t,x))\,\partial_x\lambda(t,x) + +h(x,u^*(t,x))\,\partial_{xx}\lambda(t,x) \Bigg] \; \delta_y(t,x)\,dx\,dt.
\end{aligned}
$$

As the perturbations $\delta_u(t,x)$ and $\delta_y(t,x)$ are arbitrary in the interior of $T \times X$ the functional is equal to zero if and only if equations (9.26) and (**??**).

The second functional derivative

$$
\begin{aligned}
\delta I_6^* &= \int_X \lambda(t,x)\,\delta_y(t,x)dx \Big|_{t\in\partial T} \\
&= \int_X \lambda(\bar{t},x)\,\delta_y(\bar{t},x)dx - \int_X \lambda(\underline{t},x)\,\delta_y(\underline{t},x)dx
\end{aligned}
$$

is equal to zero because $\delta_y(\underline{t}, x) = 0$ and if the transversality condition $\lambda(\bar{t}, x) = 0$ for every $x \in X$. The last two functional derivatives are

$$\begin{aligned}
\delta I_7^* &= \int_T \left[\partial_u g(x, u^*(t,x))\, y^*(t,x) + g(x, u^*(t,x))\, \delta_y(t,x) \right] \lambda(t,x)\, dt \Big|_{x \in \partial X} \\
&= \int_T \left[\partial_u g(\overline{x}, u^*(t,\overline{x}))\, y^*(t,\overline{x}) + g(\overline{x}, u^*(t,\overline{x}))\, \delta_y(t,\overline{x}) \right] \lambda(t,\overline{x})\, dt - \\
&\quad - \int_T \left[\partial_u g(\underline{x}, u^*(t,\underline{x}))\, y^*(t,\underline{x}) + g(\underline{x}, u^*(t,\underline{x}))\, \delta_y(t,\underline{x}) \right] \lambda(t,\underline{x})\, dt
\end{aligned}$$

and

$$\begin{aligned}
\delta I_8^* = \int_T \Bigg\{ &\left[\lambda(t.x)\, \partial_u \Big(\partial_x \Big(h(x, u^*(t,x)) y^*(t,x) \Big) \Big) - \partial_x \lambda(t,x)\, \partial_u h(x, u^*(t,x))\, y^*(t,x) \right] \delta_u(t,x) \\
&+ \left[\lambda(t.x)\, \partial_y \Big(\partial_x \Big(h(x, u^*(t,x)) y^*(t,x) \Big) \Big) - \partial_x \lambda(t,x)\, h(x, u^*(t,x)) \right] \delta_y(t,x) \Bigg\}\, dt \Big|_{x \in \partial X}
\end{aligned}$$

are both equal to zero because $y(t, \overline{x}) = y(t, \underline{x}) = \delta_y(t, \overline{x}) = \delta_y(t, \underline{x})0$. $\square$

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