EMA 2019-2020:
Problem set 1: linear ODE's

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15.2.2020
(revised 10.3.2020)

1 Linear scalar ODE's

1.1 Autonomous ODE's

1.1.1 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following scalar ordinary differential equations and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -\frac{1}{2}y$;

(b) $\dot{y} = \frac{1}{2}y$;

(c) $\dot{y} = 2y$;

(d) $\dot{y} = -2y$;

(e) $\dot{y} = 0$;

(f) $\dot{y} = 2$;

(g) $\dot{y} = -2$;

1.1.2 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following scalar ordinary differential equations and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -\frac{1}{2}y + 1$;

(b) $\dot{y} = \frac{1}{2}y - 1$;

(c) $\dot{y} = 2y - 2$;

(d) $\dot{y} = -2y + 2$;

(e) $\dot{y} = ay - 2$ for $a \in (-2, 2)$

(f) $\dot{y} = y + b$ for $b \in (-1, 1)$

1.1.3 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following initial value problems and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -0.5y + 1$, for $t \geq 0$ and $y(0) = 1$ for $t = 0$;

(b) $\dot{y} = 0.5y - 1$, for $t \geq 0$ and $y(0) = 1$ for $t = 0$.

1.1.4 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following terminal value problems and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -0.5y + 1$, for $t \geq 0$ and $\lim_{t\to\infty} y(t) = \overline{y}$, where $\overline{y}$ is the steady state;

(b) $\dot{y} = 0.5y - 1$, for $t \geq 0$ and $\lim_{t\to\infty} e^{-0.5t} y(t) = 0$.

1.1.5 Perform a bifurcation analysis to the following equation $\dot{y} = ay + b$ for $a \in [-2, 2]$ and $b \in (-1, 1)$.

1.1.6 Let $y = y(t)$ is a function, $y : \mathbb{R}_+ \to \mathbb{R}$. Consider the terminal value problem

$$\begin{cases} \dot{y} = gy + b & t \geq 0 \\ \lim_{t\to\infty} y(t) = \overline{y} \end{cases}$$

where $\overline{y}$ is the steady state, and g and $b \neq 0$ are constants.

(a) Assume that $g < 0$. Solve the terminal value problem and characterize the solutions analytically and geometrically.

(b) Assume that $g > 0$. Solve the terminal value problem and characterize the solutions analytically and geometrically.

1.1.7 Consider the following problem

$$\begin{cases} \dot{y} = \lambda\big(y - \bar{y}\big) & \text{for } t \in \mathbb{R}_+ \\ \int_0^\infty y(t)\,\phi(t)\,dt = \bar{y} \end{cases}$$

where $\lambda > 0$ and $\phi(t) = \lambda\, e^{-\lambda t}$. Observe that $\phi(t)$ is a distribution.

(a) Solve the problem.

(b) Provide an intuition for the problem and its solution.

1.2 Non-autonomous ODE's

1.2.1 Consider the scalar ODE

$$\dot{y} = ay + b(t),\ y : [0, \infty) \to \mathbb{R}$$

where

$$b(t) = \begin{cases} b_0 & \text{if } 0 \leq t < t^*, \\ b_1 & \text{if } t^* \leq t < \infty. \end{cases}$$

(a) Assume that $a \neq 0$ and $y(0) = y_0$ is given. Solve the initial value problem.

(b) Assume that $a > 0$ and $\lim_{t\to\infty} y(t)e^{-at} = 0$. Solve the terminal-value problem.

1.2.2 Consider the scalar ODE

$$\dot{y} = ay + b(t),\ y : [0, \infty) \to \mathbb{R}$$

where

$$b(t) = \begin{cases} b & \text{if } 0 \le t < t^*, \\ b + \Delta b & \text{if } t^* \le t < t^* + \Delta t, \\ b & \text{if } t^* + \Delta t \le t < \infty, \end{cases}$$

where $\Delta t > 0$.

(a) Assume that $a \neq 0$ and $y(0) = y_0$ is given. Solve the initial value problem.

(b) Assume that $a > 0$ and $\lim_{t\to\infty} y(t)e^{-at} = 0$. Solve the terminal-value problem.

1.2.3 Consider the scalar ODE

$$\frac{u''(x)}{u'(x)} = -\alpha,\ x \in \mathbb{R}_+,\ \alpha > 0.$$

together with the constraints

$$\alpha \int_0^\infty u'(x)dx = 1,\ \alpha\, u(0) = -1$$

(a) Prove that the solution of the problem is the constant absolute risk aversion (CARA) utility function, $u(x) = -\dfrac{-\alpha\, x}{\alpha}$

1.2.4 Consider the scalar ODE

$$\frac{y^{'}(x)\, x}{y(x)} = \mu,\ x \in \mathbb{R}$$

where μ is a constant.

(a) Prove that the general solution follows a power law.

(b) Impose conditions on the parameter and an initial value such that the solution satisfies

$$\int_{x_0}^\infty y(x)dx = 1$$

1.2.5 Consider the scalar ODE problem

$$\begin{cases} \dfrac{y^{'}(x)}{y(x)} = -x, & x \in \mathbb{R} \\ \int_{-\infty}^\infty y(x)dx = 1 \end{cases}$$

(a) Prove that the solution is the standard Gaussian probability density function $y(x) \sim N(0,1)$

1.2.6 Consider the scalar problem

$$\begin{cases} \dfrac{y'(x)}{y(x)} = -\dfrac{1+\ln(x)}{x}, & x \in (0,\infty) \\ \int_0^\infty y(x)dx = 1 \end{cases}$$

(a) Prove that the solution is the standard lognormal density function $y \sim LN(0,1)$.

1.3 Applications

1.3.1 The simplest model of population dynamics assumes that the rate of population growth is deterministic, age-independent, and constant:

$$\dot{N} = \nu N. \; N : \mathbb{R}_+ \to \mathbb{R}_+, \tag{1}$$

where $N(t)$ is the population at time t and $\nu \equiv \beta - \mu$ is the net rate of growth, β is the fertility rate and μ is the mortality rate. We assume that $N(0) = N_0 \geq 0$ is given. (References **?** see also `http://en.wikipedia.org/wiki/Exponential_growth`)

(a) Solve equation (1).

(b) Solve the initial value problem.

(c) Characterize the dynamics.

1.3.2 The stock-flow dynamics is generically represented by an equation of type,

$$\dot{A} = \pi + rA, \; A : \mathbb{R}_+ \to \mathbb{R} \tag{2}$$

where A is the stock of an asset at time t, π is net income and r is the rate of return. Assume that $r > 0$

(a) Solve equation (2) and characterise qualitatively the dynamics.

(b) Assuming we know $A(0) = A_0$, solve the initial value problem.

(c) Assuming we introduce a solvability requirement $\lim_{t\to\infty} A(t)e^{-rt} = 0$, determine the initial level of $A(0)$.

1.3.3 **?** is one of the first papers to deal with perfect foresight dynamics. The main equation of the paper was

$$\dot{p} = \beta(p - m(t)), \; p : \mathbb{R}_+ \to \mathbb{R} \tag{3}$$

where p and m are the logs of the price index and nominal money supply and $\beta > 0$

(a) Solve equation (3).

(b) Setting $p(0) = p_0$, where p_0 is known, solve the initial value problem. Does the solution to this problem makes economic sense (hint: recall the expected relationship between increases in the money supply and the price evolution) ?

(c) Let m is constant. Assuming there are no speculative bubbles, i.e, $\lim_{t\to\infty} p(t)e^{-\beta t} = 0$, determine $p(0)$.

(d) Modify the previous results assuming that there is an anticipated (to time $\tau > 0$ and finite) monetary shock.

1.3.4 The government budget constraint, in nominal variables, is

$$\dot{B} = D + iB,$$

where $B(t)$ is the stock of government bonds at time t, (where $B : \mathbb{R}_+ \to \mathbb{R}$), D is the primary deficit, and i is the interest rate on government bonds. Assume that the GDP, Y, follows the process $\dot{Y} = gY$. All variables are in nominal terms.

(a) Let $b \equiv B/Y$ and $d \equiv D/Y$. Which types of dynamic behavior for b one should expect ?

(b) Assuming we know $b(0) = b_0$, solve the initial value problem.

(c) If we introduce a solvability requirement such that $\lim_{t\to\infty} b(t)e^{-rt} = 0$, determine the initial level of $b(0)$, assuming that $r \equiv i - g > 0$.

1.3.5 Let the government budget constraint be $\dot{b} = -\tau(t) + rb(t)$ where $b(t)$ is the government debt and $\tau(t)$ is the time-varying primary surplus, at time $t \geq 0$, and $r > 0$ is the interest rate on the government debt. Assume that the government adopts a fiscal rule taking the form $\dot{\tau} = \gamma b(t) - \xi\tau(t)$ where $\gamma > 0$. Assume that the initial level of the debt is given $b(0) = b_0$.

(a) If we assume that $r > \xi$, under which conditions on the parameters of the fiscal rule can the government reach the following goal: $\lim_{t\to\infty} b(t) = 0$?

(b) Assuming the previous condition determine the paths for the government debt and primary surplus.

(c) What should be the initial surplus $\tau(0)$? Provide an intuition for this result.