AME 2020-2021:
Problem set 4: Optimal control of ODEs

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1 Calculus of variations

1. Solve the intertemporal optimization problem for a consumer, in an Arrow-Debreu exchange economy, where the utility function is $\int_0^\infty \ln{(C(t))}e^{-\rho t}dt$ where Arrow-Debreu prices and the endowment, at time t, are given by $P(t) = e^{-rt}$ and $Y(t) = Y(0)e^{\gamma t}$. The intertemporal budget constraint is $\int_0^\infty P(t)(Y(t) - C(t))dt = 0$.

   b) Find the first-order conditions for optimality (hint: this is a iso-perimetric problem).

   a) Solve the problem.

2. Consider the the intertemporal optimization problem for a consumer, in a finance economy, where the utility function is $\int_0^\infty \ln{(C(t))}e^{-\rho t}dt$. Assume that the instantaneous budget constraint, at time $t \geq 0$ is $\dot{A} = rA + Y - C$, where A is the stock of financial wealth, and r is the rate of return and Y is non-financial income. All the parameters (ρ, r, Y) are positive and constant. Let $A(0) = A_0$ and $\lim_{t\to\infty} e^{-rt}A(t) = 0$.

   a) Transform the problem into a calculus of variations problem and write the first order conditions

   b) Find the solution.

3. Assume a consumer problem in a finance economy, as in the previous example, in which the instantaneous utility function is CARA (constant absolute risk aversion) utility function. That is, consider the model:

$$\max_{C} \int_0^\infty -\frac{e^{-\theta C(t)}}{\theta} e^{-\rho t}dt,$$

   where ρ and θ are strictly positive. The restriction is $\dot{A} = rA - C$, given $A(0) = A_0 > \frac{r-\rho}{\theta r^2}$ and $\lim_{t\to\infty} e^{-rt}A(t) \geq 0$.

   a) Transform the problem into a calculus of variations problem and write the first order conditions

b) Find the solution.

4. Assume a AK model with a CARA (constant absolute risk aversion) utility function. That is, consider the model:

$$\max_{C} \int_0^{\infty} -\frac{e^{-\theta C(t)}}{\theta} e^{-\rho t} dt,$$

where ρ and θ are strictly positive, subject to the restriction $\dot{K} = AK(t) - C(t)$, given $K(0) = K_0 > \frac{A-\rho}{\theta A^2}$ and $\lim_{t\to\infty} e^{-At}K(t) \geq 0$.

   a) Transform the problem into a calculus of variations problem and write the first order conditions

   b) Find the solution.

5. Let the market value of the firm be given by the present value of cash-flows, $\int_0^{\infty} \left(Ak(t) - i(t)^2\right) e^{-rt} dt$, where k is the capital stock, i is gross investment, r is the constant rate of interest and A is a productivity parameter. The capital accumulation is characterized by the equation $\dot{k} = i - \delta k$, where $\delta > 0$ is the capital depreciation rate. The initial capital stock is, $k(0) = k_0$, is known and we require that capital is bounded asymptotically by the solvability condition $\lim_{t\to\infty} e^{-rt}k(t) \geq 0$.

   (a) Transform the problem into a calculus of variations problem. Write the optimality conditions.

   (b) Find the solution to the problem. Provide an intuition for your results.

2 Optimal control: Pontriyagin

2.1 Dynamic microeconomic models: representative consumer

1. Consider the the intertemporal optimization problem for a consumer, in a finance economy, where the utility function is $\int_0^{\infty} \ln(C(t)) e^{-\rho t} dt$. Assume that the instantaneous budget constraint, at time $t \geq 0$ is $\dot{A} = rA + Y - C$, where A is the stock of financial wealth, and r is the rate of return and Y is non-financial income. All the parameters (ρ, r, Y) are positive and constant. Let $A(0) = A_0$ and $\lim_{t\to\infty} e^{-rt}A(t) = 0$. Let $Y = ye^{(r-\rho)t}$

   a) Solve the problem using Pontriyagin's principle.

   b) Discuss the dynamic properties of the solution for different values of r.

   c) Draw the phase diagram, in (A, C)-axis, for the case $r > \rho$. Interpret the results

2. Assume a consumer problem in a finance economy, as in the previous example, in which the instantaneous utility function is CARA (constant absolute risk aversion) utility function. That is, consider the model:

$$\max_{C} \int_0^{\infty} -\frac{e^{-\theta C(t)}}{\theta} e^{-\rho t} dt,$$

where ρ and θ are strictly positive. The restriction is $\dot{A} = rA - C$, given $A(0) = A_0 > \frac{r-\rho}{\theta r^2}$ and $\lim_{t\to\infty} e^{-rt}A(t) \geq 0$.

a) determine the first order conditions by using the Pontriyagin's maximum principle;

b) prove that the solution of the system is $C(t) = C(0) + \frac{r-\rho}{\theta}t$; $A(t) = A_0 + \frac{r-\rho}{r\theta}t$, where $C(0) = rA_0 > \frac{r-\rho}{\theta r}$

2.2 Dynamic microeconomic models: representative firm

1. Let the value of the firm be given by the present value of cash-flows,

$$\int_0^{\infty} (R(K) - C(I))\, e^{-rt} dt$$

where $R(K)$ is the firm return from production and $C(I)$ is the cost of investment, K is the capital stock, I is gross investment and r is the constant rate of interest. Function $R(K)$ is increasing and concave and function $C(I)$ is increasing and convex. The capital accumulation is characterized by the equation

$$\dot{K} = I - \delta K$$

where $\delta > 0$ is the capital depreciation rate. Assume that capital is bounded asymptotically by the solvabiity condition $\lim_{t\to\infty} e^{-rt}K(t) \geq 0$.

a) Write the optimality conditions using the Pontyagin's maximum principle.

b) Provide an approximate solution to the problem both analytically and geometrically.

2. A microeconomic foundation for Tobin's Q-theory of investment (**?**). Let the value of the firm be given by the present value of cash-flows,

$$\int_0^{\infty} \left(AK^{\alpha} - I\left(1 + \frac{I}{K}\right)\right) e^{-rt} dt,\ r > 0,\ 0 < \alpha < 1,\ A > 0$$

where K is the capital stock, I is gross investment and r is the constant rate of interest. Capital accumulation is governed by

$$\dot{K} = I - \delta K$$

where $\delta > 0$ is the capital depreciation rate. Assume that capital is bounded asymptotically by the solvabiity condition $\lim_{t\to\infty} e^{-rt}K(t) \geq 0$.

a) Write the optimality conditions using the Pontyagin's maximum principle.

b) Provide an approximate solution to the problem both analytically and geometrically (hint: solve the MHDS for (Q, K) and interpret Q as Tobin's Q).

c) Write the HJB equation.

2.3 Macro and growth economics

1. Consider a version of the Ramsey model

$$\max_{C} \int_0^\infty \frac{C(t)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt,\ \sigma > 0,\ \rho > 0$$

   such that

$$\dot{K} = AK^\alpha - C(t) - \delta K,\ 0 < \alpha < 1,\ A > 0,\ \delta > 0$$

   where C and K are per capita variables.

   a) apply the Pontryiagin's principle and determine the dynamic equations which represent the first order conditions in (C, K)
   b) determine the steady states, and study their stability properties
   c) draw the phase diagram
   d) discuss the existence of a BGP, and the properties of the model regarding long run growth and transition dynamics
   e) what are the effects of an increase in productivity, A (provide both analytical and geometrical analyses).

2. Consider a version of the Ramsey model in which population varies:

$$\max_{C} \int_0^\infty \ln\left(C(t)\right) e^{nt} e^{-\rho t} dt$$

   such that

$$\dot{K} = AK^\alpha - C(t) - nK,\ 0 < \alpha < 1$$

   where C and K are per capita variables.

   a) apply the Pontryiagin's principle and determine the dynamic equations which represent the first order conditions in (C, K)
   b) determine the steady states, and study their stability properties
   c) draw the phase diagram
   d) discuss the existence of a BGP, and the properties of the model regarding long run growth and transition dynamics
   e) what are the effects of an increase in productivity, A (provide both analytical and geometrical analyses).

3. Assume a AK model with a CARA (constant absolute risk aversion) utility function. That is, consider the model:

$$\max_{C} \int_0^\infty -\frac{e^{-\theta C(t)}}{\theta} e^{-\rho t} dt,$$

   where ρ and θ are strictly positive, subject to the restriction $\dot{K} = AK(t) - C(t)$, given $K(0) = K_0 > \frac{A-\rho}{\theta A^2}$ and $\lim_{t\to\infty} e^{-At} K(t) \geq 0$.

a) determine the first order conditions, as an ode system in (C, K);

b) prove that the solution of the system is $C(t) = C(0) + \frac{A-\rho}{\theta}t$; $K(t) = K_0 + \frac{A-\rho}{A\theta}t$, where $C(0) = AK_0 > \frac{A-\rho}{\theta A}$;

c) will this model display a balanced growth path ? Discuss the properties of the model.

3 Optimal control: dynamic programming

1. Consider the intertemporal optimization problem for a consumer, in a finance economy, where the utility function is $\int_0^\infty \ln(C(t))e^{-\rho t}dt$. Assume that the instantaneous budget constraint, at time $t \geq 0$ is $\dot{A} = rA + Y - C$, where A is the stock of financial wealth, and r is the rate of return and Y is non-financial income. The parameters, ρ, r and Y are all positive constants.

   (a) Write the Hamilton-Jacobi-Bellman equation for the problem.

   (b) Solve the HJB equation, assuming that $r = \rho$.

   (c) Provide the optimal solution for $A(.)$. Interpret your results.

2. Consider the intertemporal optimization problem for a consumer, in a finance economy, where the utility function is

$$\int_0^\infty \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt.$$

   Assume that the instantaneous budget constraint, at time $t \geq 0$ is $\dot{A} = rA + Y - C$, where A is the stock of financial wealth, and r is the rate of return and Y is non-financial income. The parameters, ρ, r and the variable Y, are all positive constants.

   (a) Write the Hamilton-Jacobi-Bellman equation for the problem.

   (b) Solve the HJB equation.

   (c) Provide the optimal solution for $A(.)$. Interpret your results.

3. The Hamilton-Jacobi-Bellman equation for the intertemporal problem of a consume is

$$\rho V(a) = \max_c \left\{ U(c) + V'(a)\big(z - c\big) \right\}$$

   where c is consumption (and the control variable), a is the net wealth (and the state variable) of the consumer, $z = ra$ is the total income, and r is the interest rate.

   (a) Find implictly the optimal policy function.

   (b) Assume that the utility function is of the CRRA type, satisfying (see Problem set: non-linear ODE)

$$\frac{U'(c)}{U''(c)} = -\frac{c}{\theta}$$

Prove that $\dfrac{V'(a)}{V''(a)}$ should also be an affine function of a.

(d) From the above result prove that optimal consumption should also be an linear function of a

(d) If the total income included non-financial income, that is $z = y + ra$, will this change fundamentally change the nature of the solution ?

4. Let the value of the firm be given by the present value of cash-flows,

$$\int_0^\infty \left(AK - I^2\right) e^{-rt} dt$$

where K is the capital stock, I is gross investment and r is the constant rate of interest. The capital accumulation is characterized by the equation

$$\dot{K} = I - \delta K$$

where $\delta > 0$ is the capital depreciation rate. Assume that capital is bounded asymptotically by the solvabiity condition $\lim_{t\to\infty} e^{-rt} K(t) \geq 0$.

a) Write the optimality conditions using the principle of dynamic programming

b) Find the solution to the problem. Provide an intuition for your results