Universidade de Lisboa
Instituto Superior de Economia e Gestão

PhD in Economics
Advanced Mathematical Economics
2018-2019

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Exam: **Época Normal**
23.1.2019 (18.00h-21.00h)

Partial closed book exam: students can only use a four-page note with formulas.

1. `[2 points]` Let $y = y(t)$ is a function, $y : \mathbb{R}_+ \to \mathbb{R}$. Consider the terminal value problem

$$\begin{cases} \dot{y} = gy + b & t \geq 0 \\ \lim_{t\to\infty} y(t) = \overline{y} \end{cases}$$

where $\overline{y}$ is the steady state, and g and $b \neq 0$ are constants.

   (a) Assume that $g < 0$. Solve the terminal value problem and characterize the solutions analytically and geometrically.

   (b) Assume that $g > 0$. Solve the terminal value problem and characterize the solutions analytically and geometrically.

2. `[3 points]` Let $y = y(t)$ is a function, $y : \mathbb{R}_+ \to \mathbb{R}^2$. Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}.$$

   (a) Solve the ODE.

   (b) Draw the phase diagram and characterize it.

   (c) Let $\mathbf{y}(0) = (-1, 1)$. Solve the initial value problem.

3. `[3 points]` Consider the ODE $\dot{y} = \lambda y\,(1 - y)$, for $y : \mathbb{R}_+ \to \mathbb{R}$ and $\lambda \in \mathbb{R}$.

   (a) Find the explicit solution solution for the ODE.

   (b) Draw the bifurcation diagram. Discuss the local and global dynamic properties for the relevant cases associated to different values of λ.

4. `[4 points]` Consider a simple habit-formation model in which there are no financial restrictions. Let $c : \mathbb{R}_+ \to \mathbb{R}_+$ be the flow of consumption and $z : \mathbb{R}_+ \to \mathbb{R}_+$ be the stock of habits. The intertemporal utility functional is

$$\int_0^\infty \ln\big(c(t)\, z(t)^{-\varphi}\big) e^{-\rho t} dt,$$

where $\varphi > 1$ and $\rho > 0$, subject to the following constraints

$$\begin{cases} \dot{z} = \mu(c - z) & t \in \mathbb{R}_+ \\ z(0) = z_0 > 0 & t = 0 \\ \lim_{t\to\infty} z(t) \geq 0 \end{cases}$$

where $\mu > 0$ and z_0 is a known constant.

(a) Write the first order conditions according to the Pontriyagin's principle Are those conditions necessary and sufficient ?

(b) Write the maximized Hamiltonian dynamic system in (z, c).

(c) Find the solution to the problem (hint: define $x(t) = z(t)/c(t)$ and use the transversality condition).

(d) Provide an intuition to the solution to the problem, and, in particular to its asymptotic properties.

5. `[2 points]` Consider the first-order partial differential equation (PDE) $y_t(t, x) + ax\, y_x(t, x) = 0$, where $y = y(t, x) \in \mathbb{R}$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

   (a) Find the solution to the PDE (hint use the method of characteristics).

   (b) Let $y(0, x) = e^{-x^2}$. Find the solution to the initial-value problem. Provide an intuitive characterization of the solution.

6. `[3 points]` Consider the parabolic partial differential equation $u_t - u_{xx} = 0$, where $u = u(t, x) \in \mathbb{R}$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

   a) Find the solution to the PDE.

   b) If $u(0, x) = \delta(x - x_0)$, where $\delta(.)$ is Dirac's delta "function" and $x_0 > 0$ find the solution to the initial-value problem. Provide an intuitive characterization of the solution.

7. `[3 points]` Consider the diffusion equation

$$dX(t) = \gamma X(t)dt + \sigma dW(t)$$

   where $\{W(t)\}$ is a standard Brownian motion.

   (a) Let $X(0) = x_0$ be known. Find the solution of the initial value problem.

   (b) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

   (c) Write the forward Kolmogorov equation for the density associated to $X(t) = x$. Provide an intuition for this equation.