Universidade de Lisboa
Instituto Superior de Economia e Gestão

PhD in Economics
Advanced Mathematical Economics
2018-2019

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Exam: **Época Recurso**
5.2.2019 (18.00h-21.00h)

Partial closed book exam: students can only use a four-page note with formulas.

1. `[3 points]` The government budget constraint, in nominal variables, is

$$\dot{B} = D + iB,$$

where $B(t)$ is the stock of government bonds at time t, (where $B : \mathbb{R}_+ \to \mathbb{R}$), D is the primary deficit, and i is the interest rate on government bonds. Assume that the GDP, Y, follows the process $\dot{Y} = gY$. All variables are in nominal terms.

   (a) Let $b \equiv B/Y$ and $d \equiv D/Y$. Which types of dynamic behavior for b one should expect ?

   (b) Assuming we know $b(0) = b_0$, solve the initial value problem.

   (c) If we introduce a solvability requirement such that $\lim_{t\to\infty} b(t)e^{-rt} = 0$, determine the initial level of $b(0)$, assuming that $r \equiv i - g > 0$.

2. `[3 points]` Consider the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} a & b \\ 0 & -b \end{pmatrix}.$$

where a and b are arbitrary constants.

   (a) Which type of dynamics one would have.

   (b) Let $a + b \neq 0$. Solve the ODE.

   (c) Let a and b be strictly positive. Find, and characterize (as regards existence and uniqueness) the solutions converging asymptotically to $\mathbf{y} = (0, 0)^\top$.

3. `[3 points]` Consider the following ODE:

$$\dot{y}_1 = y_1(1 - y_2^2),$$
$$\dot{y}_2 = y_2(y_1 - 1)$$

for $(y_1, y_2) \in \mathbb{R}$.

(a) Find the equilibrium points and characterize them.

(b) Prove that the equation has a first integral: $V(y_1, y_2) = \frac{1}{2}\log(y_1^2 y_2^2) - (y_1 + \frac{1}{2}y_2^2)$. Which type of dynamics would be associated to it ?

(c) Draw the phase diagram and provide an intuition.

4. `[3 points]` Let the market value of the firm be given by the present value of cash-flows, $\int_0^\infty (Ak(t) - i(t)^2)\, e^{-rt} dt$, where k is the capital stock, i is gross investment, r is the constant rate of interest and A is a productivity parameter. The capital accumulation is characterized by the equation $\dot{k} = i - \delta k, \quad$ where $\delta > 0$ is the capital depreciation rate. The initial capital stock is, $k(0) = k_0$, is known and we require that capital is bounded asymptotically by the solvability condition $\lim_{t\to\infty} e^{-rt} k(t) \geq 0$.

   (a) Transform the problem into a calculus of variations problem. Write the optimality conditions.

   (b) Find the solution to the problem. Provide an intuition for your results.

5. `[3 points]` Consider the first-order partial differential equation $u_t(t,x) + \gamma\,(x u_x(t,x) + u(t,x)) = 0$, where $u = u(t,x)$ and $(t,x) \in (0,\infty) \times (-\infty, \infty)$, and γ is a constant.

   (a) Find the solution of the PDE (hint: use the method of characteristics).

   (b) Let $u(0,x) = e^{-\sqrt{x^2}}$. Find the solution to the initial-value problem.

6. `[3 points]` Consider the parabolic partial differential equation $u_t + u_{xx} = 0$, where $u = u(t,x) \in \mathbb{R}$ and $(t,x) \in (0,T) \times \mathbb{R}$.

   a) Find the solution to the PDE.

   b) If $u(T,x) = \delta(x)$, where $\delta(.)$ is Dirac's delta "function" , find the solution of the terminal-value problem. Provide an intuitive characterization of the solution.

7. `[2 points]` Consider the diffusion equation

$$dX(t) = (1 - X(t))dt + dW(t),\ t \in [0,\infty)$$

   where $\{W(t)\}$ is a standard Wiener process.

   (a) Let $X(0) = 0$. Find the solution of the initial value problem.

   (b) Find $\mathbb{E}[X(t)|X(0) = 0]$ and $\mathbb{V}[X(t)|X(0) = 0]$.