Universidade de Lisboa
Instituto Superior de Economia e Gestão

PhD in Economics
Advanced Mathematical Economics
2020-2021

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Exam: **Exam**(Época Normal)
18.1.2021 (18.00h-21.00h)

Instructions:

- This is an open book exam. The use of any electronic device is forbidden.
- Your answer should be concise, objective and specific. Any notation, calculation, motivation, discussion or explanation **not strictly related to the specific question** it tries to address will either not be considered or have a negative assessment.

1. `[2 points (1,1)]` Consider the initial value problem (IVP)

$$\begin{cases} \dot{y} \; + \lambda\, y = f(t), & t \in \mathbb{R}_+ \\ y(0) = 0 & t = 0 \end{cases}$$

where λ is a constant, and $f(\cdot)$ is an arbitrary function, not necessarily continuous, which is sometimes called a "driving force".

   (a) Prove that the solution to the IVP is a convolution, $y(t) = f(t) * g(t)$, where f is a driving force and g is a function sometimes called unit impulse response function (IRF).[1] Provide an intuition for this fact.

   (b) Assume that $\lambda > 0$. If we consider y represents the variation of a macroeconomic variable subject to a temporary shock

$$f(t) = \begin{cases} \alpha, & 0 \le t \le \tau \\ 0 & t > \tau, \end{cases}$$

   where $\alpha > 0$ is a constant. Find the solution to the problem. Draw the solution path.

2. `[4 points (2.5,1,0.5)]` Consider the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$ where $\mathbf{y} \in \mathbb{R}^2$

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

   (a) Solve the ODE.

   (b) Draw the phase diagram. Present the analytical expressions for the isoclines and the eigenspaces.

   (c) Assume that $y_2(0) = 0$ and that $\lim_{t\to\infty} y_1(t) = -\frac{1}{2}$. Find the particular solution.

[1] A convolution of two functions $f(x)$ and $g(x)$, for $x \in \mathbb{R}$, is a function $h = f * g$ such that $h(x) = \int_0^x f(t)\, g(x - t)\, dt$.

3. `[4 points (1,1,1.5,0.5)]` Consider the following ODE:

$$\dot{y}_1 = y_1(1 - \lambda\, y_2^2),$$
$$\dot{y}_2 = y_2(y_1 - 1)$$

for $(y_1, y_2) \in \mathbb{R}$ and λ an arbitrary real number.

(a) Find the equilibrium points and characterize them.
(b) Prove that the equation has a first integral: $V(y_1, y_2) = \frac{1}{2}\log(\lambda\, y_1^2\, y_2^2) - (y_1 + \frac{\lambda}{2} y_2^2)$. Which type of dynamics would be associated to it ?
(c) Draw the phase diagram and provide an intuition.
(d) Is λ a bifurcating parameter ? Why ?

4. `[4 points (1.5, 1.5, 0.5, 0.5)]` Consider the following problem for a small open economy

$$\max_{c} \int_0^{\infty} \ln(c(t))\, e^{-\rho\, t}\, dt, \text{ where } \rho > 0$$
$$\text{subject to}$$
$$\dot{a} \ = y - \pi(R)\, c + r\, a$$
$$a(0) = a_0 > -\frac{y}{r} \text{ given}$$
$$\lim_{t \to \infty} e^{-r\, t} a(t) \geq 0$$

where a is the net asset position of the economy, c is total consumption (of domestically produced and imported goods), y is the domestic output, r is the international interest rate and $\pi(R) \equiv \left(\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right)^{-1} R^{1-\alpha}$ is the relative price of consumption relative to domestic producer prices, R is the real exchange rate, and α is the share of home goods in total consumption. In this economy, the international interest rate is given and there is a regime of fixed exchange rates (which sporadic changes in parity).

(a) Find the optimal policy function $c^* = C(a)$, by using the principle of dynamic programming.
(b) Find the solution to that the problem. Check if the terminal condition is satisfied.
(c) Characterize the possible optimal trajectories for the economy.
(d) Assume there is a non-anticipated, permanent and constant change in R. Study the effects of this change on the economy.

5. `[6 points (1.5,1.5,0.5,2,0.5)]` Consider the following stochastic optimal control problem for a small open economy

$$\max_{C} \mathbb{E}_0\Big[\int_0^{\infty} \ln(c(t))\, e^{-\rho\, t}\, dt\Big], \text{ where } \rho > 0$$
$$\text{subject to}$$
$$dA(t) \ = \big(r\, A - \pi\, C\big)\, dt + \sigma\, A\, dW(t)$$
$$A(0) = a_0 > 0, \text{given}$$

where A is the net international asset position of the economy, C is total consumption (of domestically produced and imported goods), r and σ are the (constant) mean rate of return and volatility of international assets, and π is the (constant) relative price of consumption, and $\{W(t)\}$ is a Wiener process.

(a) Find the optimal policy function $C^* = C(A)$, by using the principle of dynamic programming.

(b) The budget constraint in the optimum is a stochastic differential equation. Find it and solve it.

(c) Let $p(t, a) = \mathbb{P}[A(t) = a | A(0) = a_0]$. Write the FPK equation for the distribution of A.

(d) Prove that

$$p(t, a) = \frac{a_0}{a\sqrt{2\pi\sigma^2 t}} \exp\left\{ -\frac{\left(\ln(a/a_0) + \frac{\sigma^2}{2} + \rho - r\right)^2}{2\sigma^2 t} \right\}$$

(e) Find $\mathbb{E}_0[A(t)]$ and $\mathbb{V}_0[A(t)]$. Discuss your results.