Universidade de Lisboa
Instituto Superior de Economia e Gestão

PhD in Economics
Advanced Mathematical Economics
2020-2021

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Exam: **Re-sit exam**(Época de Recurso)
3.2.2021 (18.00h-21.00h)

Instructions:

- This is an open book exam. The use of any electronic device is forbidden.
- Your answer should be concise, objective and specific. Any notation, calculation, motivation, discussion or explanation **not strictly related to the specific question** it tries to address will either not be considered or have a negative assessment.

1. `[3 points, (2,1) ]` A utility function, $u(x)$, is said to display constant relative risk aversion if is satisfies

$$\frac{u''(x)}{u'(x)} = -\alpha,\ x \in \mathbb{R}_+$$

where $\alpha > 0$ is called the coefficient of absolute risk aversion.

   (a) Find the general solution to the ODE.

   (b) The popular form of the CARA utility function in the literature is $u(x) = -\frac{e^{-\alpha x}}{\alpha}$. Assuming that the constraint $\alpha \int_0^\infty u'(x)dx = 1$ is satisfied, find the condition which is implicitly assumed in the previous popular form.

2. `[5 points, (2,1,2)]` Consider the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$ where $\mathbf{y} \in \mathbb{R}^2$

$$\mathbf{A} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

   (a) Find the steady state (if it exists) and characterize it.

   (b) Draw the phase diagram. Justify it analytically.

   (c) Assume that we have the initial value $\mathbf{y}(0) = (y_1(0), y_2(0))^\top = (0,1)^\top$. Find the solution to the Cauchy problem.

3. `[4 points, (0.5, 1.5, 1, 1) ]` Consider the following ODE:

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_1\,(\lambda + y_1 - y_2) \end{aligned}$$

for $(y_1, y_2) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

   (a) Find the equilibrium points.

(b) Provide a complete analytical characterization of their local dynamics, depending on the values of the parameter λ.

(c) Draw the phase diagrams of the different cases you identified in (b).

(d) Do homoclinic and/or heteroclinic exist ? Under which conditions ? Provide a geometric justification of their existence.

4. `[2 points, (1.5, 0.5)]` Consider the Cauchy problem

$$\begin{cases} \partial_t y(t,x) + \partial_x y(t,x) = e^{-\lambda t}\, y(t,x), & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ y(0,x) = e^{-|x-\mu|}, & \text{for } (t,x) \in \{t=0\} \times \mathbb{R} \end{cases}$$

where $\lambda > 0$ and μ are constants.

(a) Solve the problem.

(b) Characterize the asymptotic state, when $t \to \infty$.

5. `[6 points, (1, 2, 3)]` Consider the diffusion equation

$$dX(t) = \big(u - \sigma\, X(t)\big)\, dW(t),\; X \in \mathbb{R},\; t \in \mathbb{R}_+$$

where u and $\sigma > 0$ are known constants and $\{W(t)\}$ is a standard Brownian motion. Let $X(0) = x_0$ be a known constant.

(a) Is there an absorbing state ? If this is the case denote it by $\bar{x}$. Find the solution to the initial value problem (Hint: define $Y(t) = X(t) - \bar{x}$ and solve first the transformed problem for $Y(t)$).

(b) Write the Fokker-Planck (or forward Kolmogorov) equation for the density associated to $X(t) = x$ (or $Y(t) = y$). Solve it. Find $\mathbb{E}_0[X(t)]$ and $\mathbb{V}_0[X(t)]$.

(c) Assume we want to control the volatility of the process $\{X\}$ by solving the stochastic optimal control problem

$$\max_{U(\cdot)} \mathbb{E}_0\Big[\int_0^\infty -U(t)^2\, e^{-\rho t}\, dt\Big],\; \rho > 0$$

$$\text{subject to}$$

$$dX(t) = \big(U(t) - \sigma X(t)\big)\, dW(t),\; t \in \mathbb{R}_+$$

$$X(0) = x_0 \text{ given.}$$

Find the optimal processes $\{U^*\}$ and $\{X^*\}$ (Hint: if you use the dynamic programming principle try the value function $v(x) = \beta\, x^2$). Find $\mathbb{E}_0[X^*(t)]$ and $\mathbb{V}_0[X^*(t)]$. Compare with the non-optimal statistics in (b).