Advanced Mathematical Economics

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Contents

Chapter 7

Introduction to optimal control: the maximum principle approach

7.1 Introduction

The Pontriagin's maximum principle (PMP), see Pontryagin et al. (1962), provides first order necessary conditions for the optimal control problem, from a slightly different approach than the Calculus of Variations approach. Although the two approaches lead to the same solution for when applied to the same simple problems, the PMP can be more flexible when dealing with some type of problems (singular problems, for instance). As we will see that the first-order conditions are a system of two ordinary differential equations (and not an implicit second order ODE) it allows a more direct use of results from the theory of ODE's, and also to use geometrical methods when the constitutive functions of the problem are not completely specified or the first-order conditions do not have explicit solutions.

In this chapter we consider the same cases as we did in the calculus of variations chapter.

We denote again the independent variable by x and assume it has the domain $\mathrm{X} \subseteq \mathbb{R}$. We can write $\mathrm{X} = [x_0, x_1]$ if it is a closed set, or $\mathrm{X} = (x_0, x_1)$ if it is open, where $x_0 < x_1$ are fixed or determined optimally, and x_1 can be bounded or unbounded. In problems in which time is the independent variable we use instead t as the independent variable and $\mathrm{T} \subseteq \mathbb{R}_+$ as its domain, and we usually set $t_0 = 0$.

The optimal control problem has two variables we need to find: the **state variable**, denoted by $y(x)$ (or $y(t)$) and the **control variable**, denoted by $u(x)$ (or $u(t)$ in the time-domain). As we consider only problems in which the state variable is of dimension one, the state variable is a mapping $y : \mathrm{X} \to \mathrm{Y} \subseteq R$ and the control variable is a mapping $u : \mathrm{X} \to \mathrm{U} \subseteq R^m$. That is, we may have more m control variables. Again it is important to distinguish between the **point-wise level** of variables, i.e $y(x')$ and $u(x')$ $x = x'$, from the **curves traced out in the range of variables** $(y(x))_{x \in \mathrm{X}}$ and $(u(x))_{x \in \mathrm{X}}$, or **paths or trajectories** if the time-domain $(y(t))_{t \in \mathrm{T}}$ and

$\left(u(t)\right)_{t\in\mathrm{T}}$.

A solution to an optimal control problem allows for finding optimal curves (optimal trajectories), traced out in the specified or optimized domain, (or time interval) according to one criterium (a functional) and given some constraints. The constraints can be be specified point-wise for all the domain of the independent variable, can be specified in particular points (usually boundary points in X or T), or for all the domain (we will see this case in the next chapter).

If we can find an optimality criterium for the pointwise behavior of the state and control variables, say functions $y^*(x)$ and $u^*(x)$ for every $x \in \mathrm{X}$ (or $y^*(t)$ and $u^*(t)$ for every $t \in \mathrm{T}$) then we can trace-out the optimal curves $\left(y^*(x)\right)_{x\in\mathrm{X}}$ and $\left(u^*(x)\right)_{x\in\mathrm{X}}$.

From now on we denote the curves by $y \equiv \left(y(x)\right)_{x\in\mathrm{X}}$ and $u \equiv \left(u(x)\right)_{x\in\mathrm{X}}$.

The optimal control problem consists in finding functions $y \in \mathcal{Y}$ and $u \in \mathcal{U}$, where $\mathcal{Y} \in C^1(\mathbb{R})$, the set of continuous and continuously differentiable functions $y : \mathrm{X} \to \subseteq \mathbb{R}$, and $\mathcal{U} \in PC^1(\mathbb{R})$, the set of piecewise continuous functions $u : \mathrm{X} \to U \subseteq \mathbb{R}^m$ such that

$$y' = G(y(x), u(x), x), \text{ for } x \in [x_0, x_1] \tag{7.1}$$

that maximize the functional

$$\mathsf{J}[y, u] \equiv \int_{x_0}^{x_1} F(x, y(x), u(x))\, dx \tag{7.2}$$

with additional data on the boundaries of sets X and Y. The additional data is related to the information concerning the boundary values of the independent variable x_0 and x_1 and/or the boundary values for the state variable $y(x_0)$ and $y(x_1)$.

In most applications in macroeconomics and growth theory the independent variable is time. In this case $x = t$ and $t \in \mathrm{T} \subseteq \mathbb{R}_+$. In this case, the optimal control problem consists in finding functions $y \in \mathcal{Y}$ and $u \in \mathcal{U}$, where $\mathcal{Y} \in C^1(\mathbb{R})$, the set of continous and continuously differentiable functions $y : \mathrm{X} \to \mathrm{Y} \subseteq \mathbb{R}$, and $\mathcal{U} \in PC^1(\mathbb{R})$, the set of piecewise continuous functions $u : \mathrm{X} \to U \subseteq \mathbb{R}^m$ such that

$$\dot{y} = G(y(t), u(t), t), \text{ for } t \in [t_0, t_1] \tag{7.3}$$

that maximize the functional

$$\mathsf{J}[y, u] \equiv \int_{t_0}^{t_1} F(t, y(t), u(t)) dt \tag{7.4}$$

with additional data is given. The additional data is related to the information concerning the boundary values of the independent variable t_0 and t_1 and the boundary values for the state variable $y(t_0)$ and $y(t_1)$.

The necessary conditions for an optimum according to the **Pontriyagin's maximum principle** are set by using the **Hamiltonian** function, defined as

$$H(x, y, u, \lambda) = F(x, y, u) + \lambda\, G(x, y, u).$$

where λ, called the co-state variable, is a piecewise continuous mapping $\lambda : X \to \mathbb{R}$. When the independent variable is time, i.e, $\mathrm{X} = \mathrm{T}$, we write

$$H(t, y, u, \lambda) = F(t, y, u) + \lambda\, G(t, y, u).$$

where λ is a piecewise continuous function $\lambda : \mathrm{T} \to \mathbb{R}$.

Next we present the optimality conditions for a bounded domain, not necessarily time, in section 7.2 and an example in section **??**. Then we move to the time domain particular problems in section 7.3 and present several economic applications.

7.2 General domain problems

7.2.1 Constraints on the boundary values of the state variable

In this subsection we assume that the data of the problem includes the boundary values for the independent variable: i.e., x_0 and x_1 are known. The optimal control problem is to find an optimal control curve $(u^*(x))_{x\in[x_0,x_1]}$ that maximizes the functional (7.2) subject to ODE constraint (7.1) and, possibly additional information for the state variables at the boundary values for the independent variable.

In other words: the bounds of the domain X are known and the limits of the curves $y \in \mathrm{Y}$, traced out by $y(x)$ for $x \in \mathrm{X}$, may be known or may be chosen optimally.

Formally, the problem is

$$\begin{aligned} &\max_{u(\cdot)} \int_{x_0}^{x_1} F(x, y(x), u(x))\, dx \\ &\qquad \text{subject to} \\ &\qquad y' = G(y(x), u(x), x), \ \text{for } x \in [x_0, x_1] \\ &\qquad x_0 \text{ and } x_1 \text{ given} \\ &\qquad \text{conditions on}\, y(x_0) \text{ and } y(x_1) \end{aligned} \tag{P1}$$

We can consider the following cases:

$$\text{(a) both boundary values are known: } y(x_0) = y_0 \text{ and } \ y(x_1) = y_1 \text{ fixed;} \tag{P1a}$$

$$\text{(b) the lower boundary value is known: } y(x_0) = y_0 \text{ fixed and } \ y(x_1) \text{ free;} \tag{P1b}$$

$$\text{(c) the upper boundary value is known: } y(x_0) \text{ free and } \ y(x_1) = y_1 \text{ fixed;} \tag{P1c}$$

$$\text{(d) both boundary values are free: } y(x_0) \text{ and } \ y(x_1) \text{ free;}. \tag{P1d}$$

Proposition 1. ***[First order necessary conditions for fixed boundary values of the independent variable]*** *Let (y^*, u^*) be a solution (curve) to the OC problem* (P1) *in which one of* *the conditions* (P1a)*, or* (P1b)*, or* (P1c) *or* (P1d) *is introduced. Then there is a piecewise continuous function* $\lambda : [x_0, x_1] \to \mathbb{R}$ *, called co-state variable, such that the curves* (y^*, u^*, λ) *satisfy the following conditions:*

- *the optimality condition* [1]*:*

$$H_u(x, y^*(x), u^*(x), \lambda(x)) = 0, \text{ for each } x \in [x_0, x_1] \tag{7.6}$$

- *the multiplier equation*

$$\lambda' = -H_y(x, y^*(x), u^*(x), \lambda(x)), \text{ for each } x \in (x_0, x_1) \tag{7.7}$$

- *the constraint of the problem:*

$$y^{*'} = G(x, y^*(x), u^*(x)), \text{ for each } x \in (x_0, x_1) \tag{7.8}$$

- *and the adjoint conditions associated to the boundary conditions* (P1a) *to* (P1d)
  - *for problem* (P1a)

$$y^*(x_0) = y_0 \text{ for } x = x_0, \text{ and } y^*(x_1) = y_1 \text{ for } x = x_1, \tag{7.9}$$

  - *for problem* (P1b)

$$y^*(x_0) = y_0 \text{ for } x = x_0, \text{ and } \lambda(x_1) = 0 \text{ for } x = x_1, \tag{7.10}$$

  - *for problem* (P1c)

$$\lambda(x_0) = 0 \text{ for } x = x_0, \text{ and } y^*(x_1) = y_1 \text{ for } x = x_1, \tag{7.11}$$

  - *for problem* (P1d)

$$\lambda(x_0) = 0 \text{ for } x = x_0, \text{ and } \lambda(x_1) = 0 \text{ for } x = x_1. \tag{7.12}$$

Proof. (Heuristic) Let $u^* = (u^*(x))_{x \in \mathrm{X}}$ be an optimal control curve and let $y^* = (y^*(x))_{x \in \mathrm{X}}$ be the associated state curve. The value of the problem is

$$\mathsf{J}[y^*, u^*] = \int_{x_0}^{x_1} F(x, y^*(x), u^*(x))\, dx.$$

It is an optimiser if $\mathsf{J}[y^*, u^*] \geq \mathsf{J}[y, u]$ is satisfied for any other admissible pair of functions $(u(x), y(x))$.

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[1] We use the notation $H_u(x, y(x), u(x), \lambda(x)) \equiv \frac{\partial H(x, y(x), u(x), \lambda(x))}{\partial u}$ is the derivative evaluated at point $x \in \mathrm{X}$ for any curves (y, u, λ) and $H_u(x, y^*(x), u^*(x), \lambda(x))$ is the derivative evaluated for the optimal curves (y^*, u^*). The derivatives for y are denoted in analogous way.

It is convenient to write

$$\begin{aligned}
\mathsf{J}[y^*,u^*] &= \int_{x_0}^{x_1} F(x,y^*(x),u^*(x))dx = \\
&= \int_{x_0}^{x_1} \Big[F(x,y^*(x),u^*(x)) + \lambda(x)(G(x,y^*(x),u^*(x)) - y^{*'}(x))\Big]\ dx = \\
&= \int_{x_0}^{x_1} \Big[H\Big(x,y^*(x),u^*(x),\lambda(x)\Big) - y^{*'}(x)\lambda(x)\Big]\ dx
\end{aligned}$$

Again we introduce a perturbation on the optimal state-control pair $(y,u) = (y^*,u^*)+\varepsilon\,\boldsymbol{\eta}$, where ε is a constant and $\boldsymbol{\eta} = (\eta_y,\eta_u)$. The admissible perturbations differ for the different versions of the problem: for (P1a) we should have $\eta_y(x_0) = \eta_y(x_1) = 0$, for (P1b) we should have $\eta_y(x_0) = 0$ and $\eta_y(x_1) \neq 0$, for (P1c) we should have $\eta_y(x_0) \neq 0$ and $\eta_y(x_1) = 0$, and for (P1d) we should have $\eta_y(x_0) \neq 0$ and $\eta_y(x_1) \neq 0$.

The first-order Taylor approximation of the functional is

$$\mathsf{J}[y,u] = \mathsf{J}[y^*,u^*] + \delta\mathsf{J}[y^*,u^*](\ )\epsilon + o(\epsilon)$$

where

$$\begin{aligned}
\delta_{\boldsymbol{\eta}(\cdot)}\mathsf{J}[y^*,u^*](\ ) = &\int_{x_0}^{x_1} \Big\{\ H_u(x,y^*(x),u^*(x),\lambda(x))\,\eta_u(x) + H_y(x,y^*(x),u^*(x),\lambda(x))\,\eta_y(x) - \lambda(x)\eta_y'(x)\ \Big\}dx = \\
= &\int_{x_0}^{x_1} \Big\{H_u(x,y^*(x),u^*(x),\lambda(x))\,\eta_u(x) + \left(H_y(x,y^*(x),u^*(x),\lambda(x)) + \lambda'(x)\right)\eta_y(x)\Big\}\ dx + \\
&+ \lambda(x_0)\eta_y(x_0) - \lambda(x_1)\eta_y(x_1).
\end{aligned}$$

Then $\mathsf{J}[y,u] \leq \mathsf{J}[y^*,u^*]$ only if $\delta_{\boldsymbol{\eta}(\cdot)}\mathsf{J}[y^*,u^*](\ ) = 0$, which, using similar arguments as in the case of the calculus of variations problem, is equivalent to the Pontriyagin's conditions: $H_u(.) = \lambda' - H_y(.) = 0$. The adjoint constraints should verify $\lambda(x_0)\eta_y(x_0) = \lambda(x_1)\eta_y(x_1) = 0$. From this and the admissibility values for $\eta_y(x_0)$ and $\eta_y(x_1)$ then the adjoint constraints are as in equations (7.9) to (7.12) □

7.2.2 Constraints on the boundary values of the independent variable

In this subsection we consider the case in which one or both bounds in the domain of independent variables can be optimally chosen, i.e $x \in \mathrm{X}^* = [x_0^*, x_1^*]$, where one or both x_j^*, for $j = 0, 1$ are free, but the boundary values for the state variable are fixed: i.e. $y(x_0^*) = y_0$ and/or $y(x_1^*) = y_1$ are fixed. The optimal control problem is to find the optimal cut-off values for the indepdendent variable, x_0^* and/or x_1^* and an optimal control $\left(u^*(x)\right)_{x\in[x_0^*,x_1^*]}$ that maximizes the functional (7.2) subject to ODE constraint (7.1).

In other words: the bounds of the domain X^* can be known or can be chosen optimally while the limits of the curves $y \in \mathrm{Y}$, traced out by $y(x)$ for $x \in \mathrm{X}^*$ are known.

Formally, the problem is

$$\begin{aligned} &\max_{u(\cdot)} \int_{x_0}^{x_1} F(x, y(x), u(x))\, dx \\ &\qquad\text{subject to} \\ &\qquad y' = G(y(x), u(x), x), \text{ for } x \in [x_0, x_1] \\ &\qquad y(x_0) = y_0 \text{ and } y(x_1) = x_1 \text{ given} \\ &\qquad \text{conditions on}\, x_0 \text{ and } x_1 \end{aligned} \tag{P2}$$

We can consider the following cases:

$$\text{(a) both cut-offs are known: } x_0 \text{ and } \ x_1 \text{ fixed;} \tag{P2a}$$

$$\text{(b) the lower cut-off is known: } x_0 \text{ fixed } \text{ and } \ x_1 \text{ free;} \tag{P2b}$$

$$\text{(c) the upper cut-off is known: } x_1 \text{ free } \text{ and } \ x_1 \text{ fixed;} \tag{P2c}$$

$$\text{(d) both cut-offs are free: } x_0 \text{ and } \ x_1 \text{ free;}\,. \tag{P2d}$$

Proposition 2 (First order necessary conditions for free domain and fixed boundary state variable optimal control problems). *Let* (y^*, u^*) *be a solution curve to the OC problem* (P2) *where* $y(x_0) = y_0$ *and* $y(x_1) = y_1$ *are fixed. Then there is an optimal domain for the independent variable* $x^* = [x_0^*, x_1^*] \subset \mathbb{R}$*, a piecewise continuous function* $\lambda : x^* \to \mathbb{R}$ *, called co-state variable, such that* (y^*, u^*, λ) *satisfy the optimality condition* (7.6)*, the multiplier equation* (7.7) *and the ODE constraint of the problem* (7.8)*, all for* $x \in Int(\mathrm{X}^*)$ *and the adjoint conditions associated to the boundary conditions* (P2a) *to* (P2d)

- *for problem* (P2a) $y^*(x_0) = y_0$ *and* $y^*(x_1^*) = y_1$ *and* x_0 *and* x_1 *are fixed;*
- *for problem* (P2b) $y^*(x_0) = y_0$ *and* $y^*(x_1^*) = y_1$ *and*
$$x_0^* = x_0 \text{ and } \ H(x_1^*, y_1, u^*(x_1^*)) - {y^*}'(x_1^*)\lambda(x_1^*) = 0; \tag{7.14}$$
- *for problem* (P2c) $y^*(x_0^*) = y_0$ *and* $y^*(x_1) = y_1$ *and*
$$H(x_0^*, y_0, u^*(x_0^*)) - {y^*}'(x_0^*)\lambda(x_0^*) = 0 \text{ and } \ x_1^* = x_1; \tag{7.15}$$
- *for problem* (P2d) $y^*(x_0^*) = y_0$ *and* $y^*(x_1^*) = y_1$ *and*
$$H(x_0^*, y_0, u^*(x_0^*)) - {y^*}'(x_0^*)\lambda(x_0^*) = 0 \text{ and } \ H(x_1^*, y_1, u^*(x_1^*)) - {y^*}'(x_1^*)\lambda(x_1^*) = 0. \tag{7.16}$$

Proof. Using the same method for finding perturbations we used in the proof of propositions **??** and 1, the Gâteaux derivative of the value functional,

$$\begin{aligned} \delta_{(\eta(\cdot),\chi)}\mathsf{J}[y^*, u^*; x^*] = &\int_{x_0^*}^{x_1^*} \big(H_u(x, y^*(x), u^*(x), \lambda(x))\eta_u(x) + H_y(x, y^*(x), u^*(x), \lambda(x))\eta_y(x) - \lambda(x)\eta_y'(x)\big)\, dx + \\ &+ H(x, y^*(x), u^*(x), \lambda(x))\big|_{x=x_1^*}\chi_1 - H(x, y^*(x), u^*(x), \lambda(x))\big|_{x=x_0^*}\chi_0, \end{aligned}$$

recall that we introduced the perturbation $x_j = x_j^* + \varepsilon\,\chi_j$ for $j \in \{0,1\}$. Setting $H^*(x) = H(x, y^*(x), u^*(x), \lambda(x))$, integrating by parts,

$$\begin{aligned}\delta_{(\eta(\cdot),\chi)}\mathsf{J}[y^*, u^*; x^*] \; &= \int_{x_0^*}^{x_1^*} \left(H_u^*(x)\eta_u(x) + \left(H_y^*(x) + \lambda'(x)\right)\eta_y(x)\right) dt + \lambda(x_0^*)\eta_y(x_0^*) - \lambda(x_1^*)\eta_y(x_1^*) + \\ &+ H^*(x_1^*)\chi_1 - H^*(x_0^*)\chi_0.\end{aligned}$$

Using the same approximation as in the proof of Proposition **??** yields the analogue to equation (**??**)

$$\begin{aligned}\delta\mathsf{J}[y^*, u^*; x^*]\;(\eta,\chi) = &\int_{x_0^*}^{x_1^*} \left(H_u^*(x)\eta_u(x) + \left(H_y^*(x) + \lambda'(x)\right)\eta_y(x)\right) dt + \lambda(x_0^*)\eta_0 - \lambda(x_1^*)\eta_1 + \\ &+ \left(H^*(x_1^*) - y^{*'}(x_1^*)\lambda(x_1^*)\right)\chi_1 - \left(H^*(x_0^*) - y^{*'}(x_0^*)\lambda(x_0^*)\right)\chi_0\end{aligned} \tag{7.17}$$

The adjoint necessary conditions for the optimum, because $\eta_1 = \eta_0 = 0$, are presented, for the different versions of the problem, in equations (7.14) to (7.16). □

7.2.3 A taxonomy for optimal control problems

This is a general case that encompasses combinations of all the previous cases: we assume both the domains of the independent variables and the boundary values of the state variables are free. That is x_0 and/or x_1 can be fixed or free and $y(x_0)$ and/or $y(x_1)$ can be fixed or free.

When there is a free boundary condition, for the independent variable x or for the state variable $y(x)$, it should be optimized. In the first case, the optimal control problem is to find the optimal cut-off values for the indepdendent variable, x_0^* and/or x_1^* and an optimal control $(u^*(x))_{t\in[x_0^*,x_1^*]}$ that maximizes the functional (7.2) subject to ODE constraint (**??**) and having fixed or free boundary values for the state variable. In the second case, the optimal control problem is to find the optimal boundary values for the state variable , $y^*(x_0)$ and/or $y^*(x_1)$ and an optimal control $(u^*(x))_{t\in[x_0,x_1]}$ that maximizes the functional (7.2) subject to ODE constraint (**??**) and having fixed or free cut-off values for the independent variable.

The necessary conditions include the optimality condition (7.6), the multiplier equation (7.7) and the ODE constraint of the problem (7.8), all for $x \in \text{int}(x^*)$. To get the adjoint condition associated to the terminal values of the state variable, when they need to be optimized, are obtained by setting in equation (7.17), $\eta_0 \neq 0$ and $\eta_1 \neq 0$. Therefore, the adjoint condition associated to $y^*(x_j^*)$ and $\lambda(x_j^*) = 0$, implying that the adjoint condition associated to the optimal boundary value of the independent variable, x_j^* is $H^*(x_j^*) = 0$, for $j = 0, 1$.

The adjoint conditions presented in Table 7.1 cover the 16 possible cases and it is the analogue to Table **??** for the calculus of variations problem.

Table 7.1: Adjoint conditions for bounded domain OC problems

data		optimum	
x_j	$y(x_j)$	x_j^*	$y^*(x_j^*)$
fixed	fixed	x_j	y_j
fixed	free	x_j	$\lambda(x_j) = 0$
free	fixed	$H(x_j^*, y_j, u^*(x_j^*)) - y^{*'}(x_j^*)\lambda(x_j^*) = 0$	y_j
free	free	$H(x_j^*, y^*(x_j^*), u^*(x_j^*)) = 0$	$\lambda(x_j^*) = 0$

The index refers to the lower boundary when $j = 0$ and to the upper boundary when $j = 1$

7.3 Time domain problems

Next we present two problems which are common when the independent variable is time: the constrained terminal state problem and the discounted infinite horizon problem. While the second is typical from time-domain problems, the first can also occur in general x-domain problems. If this is the case we can simply adapt the results from the previous section.

In both problem we take already presented objective functional and dynamic constraint, in equations (7.4) and (7.3.1), respectively.

7.3.1 Constrained terminal state problem

A common problem in macroeconomics is the following: the set of independent variables is known such as $t_0 = 0$ and $t_1 = \bar{t}$, the initial value of the state value is fixed, $y(0) = y_0$, the structure of the economy given by the ODE (7.4), the value functional is , and we assume that the terminal value for the state variable is constrained by $R(\bar{t}, y(\bar{t})) \geq 0$ where $y(\bar{t})$ is free. Our goal is to determine the optimal trajectories for the state variable y^* $\left(y^*(t)\right)_{t\in\mathrm{T}}$ and the u^* $\left(u^*(t)\right)_{t\in\mathrm{T}}$.

Formally the problem is

$$\begin{aligned} \max_{u(\cdot)} \int_0^{\bar{t}} & F(t, u(t), y(t))\, dt \\ & \text{subject to} \\ & \dot{y} = G(t, u(t), y), \text{ for } t \in \mathrm{T} \\ & \bar{t} \text{ given} \\ & y(0) = y_0 \text{ fixed} \\ & R(\bar{t}, y(\bar{t})) \geq 0, \end{aligned} \tag{Pt1}$$

where $\mathrm{T} = [0, \bar{t}]$ and functions $R(\cdot)$, $F(\cdot)$, and $G(\cdot)$ are known.

The Hamiltonian function is

$$H(t) = H(t, u, y, \lambda) \equiv F(t, u, y) + \lambda\, G(t, u, y), \text{ for } t \in \mathrm{T}$$

where λ is called co-state (or adjoint) and is a piecewise continuous function $\lambda : \mathrm{T} \to \mathbb{R}$. We assume H to be continuous, and continuously differentiable, and except otherwise mentioned that $H_{uu}(t) \neq 0$ for every $t \in \mathrm{T}$.

Proposition 3. ***1st order necessary conditions for the constrained terminal value problem*** *Let (y^*, u^*) be the solution trajectories for problem Pt1. Then it satisfies*

- *the optimality condition*

$$H_u(t, u^*, (t)y^*(t), \lambda(t)) = 0, \text{ for each } t \in \mathrm{T}; \tag{7.18}$$

- *the multiplier equation*

$$\dot{\lambda} = \; - H_y(t, u^*(t), y^*(t), \lambda(t)) = 0, \text{ for each } t \in \mathrm{T}; \tag{7.19}$$

- *the transversality condition*

$$\lambda(\bar{t})R(\bar{t}, y^*(\bar{t})) = 0, \tag{7.20}$$

and the admissibility conditions:

$$\dot{y}^* \; = \; G(t, u^*(t), y^*(t)) = 0, \text{ for each } t \in \mathrm{T}; \tag{7.21}$$

and

$$y^*(0) = y_0, \text{ for } t = 0 \tag{7.22}$$

Proof. In this case the value at the optimum is

$$\mathsf{J}[y^*, u^*] = \int_0^{\bar{t}} \left(H(t, u^*(t), y^*(t), \lambda(t)) - \dot{y}^*(t)\lambda(t)\right) \, dt + \psi R(\bar{t}, y(\bar{t}))$$

where ψ is a Lagrange multiplier. The functional derivative, for an arbitrary perturbation $(\delta y, \delta u) = \varepsilon(\eta_y, \eta_u)$ around (y^*, u^*), is now

$$\begin{aligned}\delta_{(\eta_y(\cdot),\eta_u(\cdot))}\mathsf{J}[y^*, u^*] = & \int_0^{\bar{t}} \left[H_u(t, y^*(t), u^*(t), \lambda(t))\eta_u(t) + (H_y(t, y^*(t), u^*(t), \lambda(t)) + \dot{\lambda}(t))\eta_y(t)\right] dt + \\ & + \lambda(0)\eta_y(0) + \left(\psi R_y(\bar{t}, y^*(\bar{t})) - \lambda(\bar{t})\right) \eta_y(\bar{t}),\end{aligned}$$

where admissible perturbations satisfy $\eta_y(0) = 0$ and $\eta_y(\bar{t}) \neq 0$. Given the inequality constraint, the KKT conditions

$$R(\bar{t}, y^*(\bar{t})) \geq 0, \; \psi \geq 0, \; \psi R(\bar{t}, y^*(\bar{t})) = 0,$$

are also necessary for an optimum. Setting $H_u^*(t) = \dot{\lambda}\,(t) - H_y^*(t) = \eta_y(0) = 0$, as presented in conditions (7.18)-(7.19). At last, because $\eta_y(\bar{t}) \neq 0$, the remaining necessary condition for an optimum is $\psi R_y(\bar{t}, y^*(\bar{t})) - \lambda(\bar{t}) = 0$. Multiplying both terms by $R(\bar{t}, y^*(\bar{t}))$ and using the KKT condition yields condition (7.20). □

7.3.2 Infinite horizon problems

Necessary conditions for an optimum

The benchmark problem in macroeconomics and growth theory is the (autonomous) **discounted infinite horizon problem** is

$$\begin{aligned} &\max_{u(\cdot)} \int_0^{\infty} e^{-\rho t} f(y(t), u(t)) dt \\ &\text{subject to} \\ &\dot{y} = g(y, u), \text{ for } \ t \in \mathrm{T} \\ &y(0) = y_0 \text{ fixed} \\ &\text{boundary conditions at infinity.} \end{aligned} \tag{Pt2}$$

Two versions, related to different boundary conditions, are usually considered

$$\lim_{t \to \infty} y(t) \text{ is free} \tag{Pt2a}$$

$$\lim_{t \to \infty} R(t, y(t)) \geq 0 \tag{Pt2b}$$

where $\rho > 0$, and function $R(t, y)$ is known and takes the form of a solvability or sustainability condition. Observe that the utility function is $F(t, y(t), u(t)) \equiv e^{-\rho t} f(y(t), u(t))$ is directly dependent on time by a discount factor, which is a bounded function of time, and we consider a version of the problem in which the constraint ODE os autonomous.

For discounted optimal control problems define the **current-value Hamiltonian** function

$$\begin{aligned} h(y(t), u(t), q(t)) &= f(y(t), u(t)) + q(t)\, g(y(t), u(t)) = \\ &= e^{-\rho t}\, H(t, y(t), u(t), \lambda(t)). \end{aligned}$$

where $q(t) = e^{\rho t}\lambda(t)$ is the current-value co-state variable. Consistently with the previous definitions we call **discounted Hamiltonian** and **discounted co-state variable** to $H(t, y, u, \lambda)$ and λ, respectively. Again q (or λ) are piecewise continuous functions $q : \mathrm{T} \to \mathbb{R}$ (or $\lambda : \mathrm{T} \to \mathbb{R}$).

Observe the current-value Hamiltonian is time-independent. If the constraint is time-dependen, i.e, if $g(t, y, u)$ then Hamiltonian is also explicitly time dependent

$$h(t, y(t), u(t), q(t)) = f(y(t), u(t)) + q(t)\, g(t, y(t), u(t)).$$

Proposition 4 (**First order necessary conditions: Pontriyagin maximum principle**). *Let (y^*, u^*) be the optimal state and control trajectory pair. Then there is a PC^1 continuous co-state variable q such that the following conditions hold:*

- *the optimality condition*

$$h_u(y^*(t), u^*(t), q(t)) = 0, \ \textit{for each} \ \ t \in [0, \infty) \tag{7.24}$$

- *the multipliers equation for the current co-state variable (also called adjoint equation)*

$$\dot{q} = \rho q - h_y(y^*(t), u^*(t), q(t)), \ \textit{for each} \ \ t \in [0, \infty) \tag{7.25}$$

- *the transversality condition or* (Pt2b)*:*
  - *associated to* (Pt2a)

$$\lim_{t \to \infty} e^{-\rho t} q(t) = 0, \tag{7.26}$$

  - *associated to* (Pt2b)

$$\lim_{t \to \infty} e^{-\rho t} q(t) y^*(t) = 0; \tag{7.27}$$

- *and the admissibility conditions*

$$\dot{y}^* = g(y^*(t), u^*(t)), \ \textit{for each} \ \ t \in [0, \infty) \tag{7.28a}$$

$$y^*(0) = y_0, \ \textit{for} \ \ t = 0. \tag{7.28b}$$

Proof. We can see this proposition as a particular case of Propositions 1 and Proposition 3. □

Remark: if the constraint ODE is non-autonomous, i.e, if $\dot{y} = g(t, y(t), u(t))$ equations (7.25) and (7.28a) will become non-autonomous ODE's, because the current value Hamiltonian becomes (directly) time-dependent, i.e. $h(t) = h(t, y(t), u(t), q(t)) = f(y(t), u(t)) + q(t)\, g(t, y(t), u(t))$.

Sufficient conditions for an optimum

The nature of the transversality condition (7.27) is a difficult technical issue associated with the solution to infinite-horizon optimal control problems (see Michel (1982) and Kamihigashi (2001)).

From now on, we assume the **Arrow sufficiency condition**: $h^*_{uu} = h(y^*, u^*, q) \leq 0$. This condition guarantees that the first-order necessary conditions for an extremum, presented in Proposition 4, are also necessary.

7.4 The dynamics of optimal control problems

The infinite-horizon discounted problem is the central structure to macroeconomics and growth theory since the 1960's.

In several applications the constitutive functions $f(y, u)$ and $g(y, u)$ are specified explicitly, or, if they have a non-linear structure the resulting system of ODE's usually cannot be solved explicitly. However, we can have a geometric interpretation for the solution of an optimal control problem in regular cases.

Writing

$$h(y, u, q) = f(y, u) + q\, g(y, u),$$

the necessary (and possibly sufficient as well) conditions for the infinite-horizon discounted optimal control problem, presented in Proposition 4, can be compactly presented as a differential-algebraic system:

$$\begin{aligned}\dot{y} &= g(y,u)\\ \dot{q} &= \rho q - h_y(y,u,q).\\ 0 &= h_u(y,u,q)\end{aligned} \tag{7.29}$$

Observe that $h_q(y,u,q) = g(y,u)$, $h_{qy}(y,u,q) = h_{yq}(y,u,q) = g_y(y,u)$, and $h_{qu}(y,u,q) = h_{uq}(y,u,q) = g_u(y,u)$.

The (local) existence and uniqueness of a steady state can be assessed from the Jacobian of system (7.29) evaluated at $\dot{y} = \dot{q} = 0$, that if

$$\mathbf{F}\ (y,u,q) = \begin{pmatrix} g_y(y,u) & g_u(y,u) & 0\\ -h_{yy}(y,u,q) & -h_{uy}(y,u,q) & \rho - g_y(y,u)\\ h_{yu}(y,u,q) & h_{uu}(y,u,q) & g_u(y,u)\end{pmatrix}.$$

The determinant of $\mathbf{F}$ is

$$\det(\mathbf{F})(y,u,q) = (g_y - \rho)\,(g_y\,h_{uu} - g_u\,h_{uy}) + g_u\,(g_u\,h_{yy} - g_y\,h_{yu})$$

where all the partial derivatives are evaluated at an arbitrary point (y,u,q). As $h_{yy}(y,u,q) = f_{yy}(y,u) + q\,g_{yy}(y,u)$, and if the functions are continuous $h_{yu}(y,u,q) = h_{yu}(y,u,q) = f_{yu}(y,u) + q\,g_{yu}(y,u) = f_{uy}(y,u) + q\,g_{uy}(y,u)$ and $h_{uu}(y,u,q) = f_{uu}(y,u) + q\,g_{uu}(y,u)$, we readily see that:

1. if $f(y,u)$ and $g(y,u)$ are linear functions both in the state and the control variable, that is in (y,u), all the second derivatives of $h_(\cdot,q)$ are equal to zero, then $\det(\mathbf{F})(y,u,q) = 0$, which implies that either a steady state does not exist, or there is an infinite number of steady states (and possibly a solution to the problem does not exist);

2. if h is linear in the control variable, that is if $h_{uu} = h_{yu} = h_{uy} = 0$ then $\det(\mathbf{F}) = -(g_u)^2\,h_{yy}$, where g_u is a constant, and a steady state can exist, if $h_{yy} \neq 0$;

3. in several models applied to macroeconomics, the objective function is independent of the state variable, that is $f = f(u)$ wthich implies $h_{uy} = q\,g_{uy}$ and $h_{yy} = 0$ In this case, we have

   $$\det(\mathbf{F}) = (g_y - \rho)\,(g_y\,h_{uu} - g_u\,q\,g_{uy}) - g_u\,g_y\,q\,g_{yu}$$

   which does not rule out the existence of a steady state. However, if the constraint function is linear in u we cannot rule out that $\det(\mathbf{F}) = 0$. Indeed, this is the case in simple endogenous growth models and simple models for the representative household.

4. if functions $f(yu)$ and $g(y,u)$ are nonlinear, having locally $\det(\mathbf{F})(y,u,q) = 0$ provides a necessary condition for the existence of local bifurcations and of potentially complex dynamics in the solution of the optimal control problem. The number of cases is potentially enormous

and complex. In abstract we can say that any bounded trajectory converging to a steady state or a limit cycle is a candidate for optimality, although we cannot rule out the possibility of existence of multiple solutions.

If functions $f(.)$ and $g(.)$ are sufficiently smooth we may qualitative characterize the optimal dynamics of (y, q) (or for (y, u)).

The algebraic equation in system (7.29) allows us to determine uniquely the control variable. If $\partial^2 h/\partial u^2 \neq 0$, the implicit function theorem allows for obtaining from the optimality condition for u, $h_u(u, y, q) = 0$, an implicit representation of the control as a function of the state and co-state variables $u = U(y, q)$.

Assume that $h_{uu}(y, u, q) \neq 0$ for any $(y, u, q) \in \mathrm{Y} \times \mathrm{U} \times \mathrm{Q}$. Then, from the implicit function theorem, we can find uniquely the control as a function of the state and the co-state variables

$$u = U(y, q)$$

where, from the implicit function theorem

$$U_y = -\frac{h_{uy}}{h_{uu}}, \text{ and } U_q = -\frac{g_u}{h_{uu}}.$$

If we substitute this control representation in the differential equations of (7.29) we obtain the **modified Hamiltonian dynamic system** (MHDS) as a non-linear planar ODE,

$$\begin{pmatrix} \dot{y} \\ \dot{q} \end{pmatrix} = \mathbf{M}\,(y, q) \equiv \begin{pmatrix} g(y, U(y, q)) \\ \rho\, q - h_y(y, U(y, q), q) \end{pmatrix}. \tag{7.30}$$

The Jacobian of the MHDS is

$$\begin{aligned} D\mathbf{M}\,(y, q) &= \begin{pmatrix} \dfrac{\partial \dot{y}(y, q)}{\partial y} & \dfrac{\partial \dot{y}(y, q)}{\partial q} \\ \dfrac{\partial \dot{q}(y, q)}{\partial y} & \dfrac{\partial \dot{q}(y, q)}{\partial q} \end{pmatrix} \\ &= \begin{pmatrix} g_y(y, q) + g_u(y, q)\, U_y(y, q) & g_u(y, q)\, U_q(y, q) \\ -h_{yy}(y, q) - h_{yu}(y, q)\, U_y(y, q) & \rho - h_{yu}(y, q)\, U_q(y, q) - h_{yq}(y, q) \end{pmatrix}. \end{aligned}$$

Lemma 1. *Let $\rho > 0$ and assume that $h_{uu}(y, u, q) \neq 0$ for any $(y, u, q) \in \mathrm{Y} \times \mathrm{U} \times \mathrm{Q}$. Then the Jacobian $D\mathbf{M}\,(y, q)$ has trace$D\,\mathbf{M}\,(y, q) = \rho > 0$ for any point $(y, q) \in \mathrm{Y} \times \mathrm{Q}$.*

Proof. Has $g_u\, U_y = -g_u\, \dfrac{h_{uy}}{h_{uu}} = h_{uy}\; U_q = -h_{uy}\; \dfrac{g_u}{h_{uu}}$, we readily see that the Jacobian matrix has the following

$$D\mathbf{M}\,(y, q) = \begin{pmatrix} M_{11}(y, q) & M_{12}\,(y, q) \\ M_{21}(y, q) & \rho - M_{11}(y, q) \end{pmatrix}$$

Therefore trace $D\mathbf{M}\,(y, q) = M_{11} + \rho - M_{11} = \rho > 0$ □

Observe that this result has a global nature. The only requirement is that $h_{uu} \neq 0$ globally. This assures that there is steady state and that there are no singularities [2]

Assume the MHDS has, at leat, one steady state, $(\bar{y}, \bar{q}) = \{(y, q) : \dot{y} = \dot{q} = 0\}$. In the neighbourhood of $(\bar{y}, \bar{q})$ we can approximate the non-linear MHDS (7.30) by the linear system

$$\begin{pmatrix} \dot{y}(t) \\ \dot{q}(t) \end{pmatrix} = D_{(y,q)}\mathbf{M}(\bar{y}, \bar{q}) \begin{pmatrix} y(t) - \bar{y} \\ q(t) - \bar{q} \end{pmatrix}$$

where the Jacobian, evaluated at the steady state $(\bar{y}, \bar{q})$ is the matrix of constants

$$D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q}) = \begin{pmatrix} \dfrac{\partial \dot{y}(\bar{y}, \bar{q})}{\partial y} & \dfrac{\partial \dot{y}(\bar{y}, \bar{q})}{\partial q} \\ \dfrac{\partial \dot{q}(\bar{y}, \bar{q})}{\partial y} & \dfrac{\partial \dot{q}(\bar{y}, \bar{q})}{\partial q} \end{pmatrix}.$$

If functions $f(.)$ and $g(.)$ have no singularities we can obtain a generic characterization of the dynamics of the MHDS, and, therefore, of the solution to the optimal control problem.

Proposition 5. *Let there be a steady state* $(\bar{y}, \bar{q})$ *for the MHDS system. This steady state can never be locally a stable node or focus. There is transitional dynamics converging to the steady state only if it is a saddle point, that is if* $\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) < 0$.

Proof. Evaluating the Jacobian at a steady state $(\bar{y}, \bar{q})$, with $\bar{u} = u(\bar{y}, \bar{q})$, we find that the Jacobian matrix becomes a matrix of constants

$$D_{(y,q)}\mathbf{M}\ (\bar{y}, \bar{q}) = \begin{pmatrix} \bar{g}_y - \dfrac{\bar{g}_u \bar{h}_{uy}}{\bar{h}_{uu}} & -\dfrac{(\bar{g}_u)^2}{\bar{h}_{uu}} \\ -\bar{h}_{yy} + \dfrac{(\bar{h}_{uy})^2}{\bar{h}_{uu}} & \rho - \bar{g}_y + \dfrac{\bar{g}_u \bar{h}_{uy}}{\bar{h}_{uu}} \end{pmatrix}$$

where $\bar{g}_y = g(u(\bar{y}, \bar{q}), \bar{y})$, etc[3]. Observe that the Jacobian matrix has a particular structure

$$D_{(y,q)}\mathbf{M}(\bar{y}, \bar{q}) = \begin{pmatrix} a & b \\ c & \rho - a \end{pmatrix}. \tag{7.31}$$

impiying that the trace is equal to the rate of time preference,

$$\text{trace}\ (D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})) = \rho > 0 \tag{7.32}$$

and is always positive and

$$\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) = a(\rho - a) - bc. \tag{7.33}$$

This implies that, if there is a steady state, it can never a stable node or focus. Therefore, it can be an unstable node or focus if $\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) > 0$, a saddle-point if $\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) < 0$ or a degenerate saddle node if $\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) = 0$. There can only be transitional dynamics if it is a saddle-point. □

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[2] In **?** we deal with the case in which we can have locally $h_{uu} = 0$. This tends to make the solution to be non-unique locally for a subset of the space Y × Q.

[3] because if $h(.)$ is continuous then $h_{uy}(.) = h_{yu}(.)$.

Then we can conclude the following:

1. in generic cases the equilibrium point $(\bar{y}, \bar{q})$ is a saddle point if $\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) < 0$, and:
    (a) the stable manifold associated with $(\bar{y}, \bar{q})$
    $$W^s = \{\ (y, q) \in \mathrm{Y} \times \mathrm{Q}\ \subseteq \mathbb{R}^2 : \lim_{t \to}(y(t), q(t)) = (\bar{y}, \bar{q})\}$$
    passing through point $y(0) = y_0$ **is the solution set of the OC problem**;
    (b) the solution to the OC problem is (at least locally) unique;
    (c) the optimal trajectory is asymptotically tangent to the stable eigenspace E^s associated to Jacobian $D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})$

2. if $\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) > 0$ the equilibrium point is unstable. In this case the candidate solutions tend to diverge away from the steady state. The existence of a solution depends on the satisfaction of the transversality condition (equation (7.28b)). If this is the case, the solution to the optimal control problem is **non-stationary**, that is, it will be asymptotically unbounded;

3. if $\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) = 0$, there will be an infinite number of candidate steady states. The system is "anchored" however by the initial condition $y(0) = y_0$, which means that the solution to the optimal control problem exists, it is unique, but it is **stationary**;

A relatively common problem in economics, involve functions $f(y, u)$ and $g(y, u)$, in which g is a linear function and f is independent from the state variable, that is $f = f(u)$ with $f_{uu} < 0$ and $g(y, u) = a\,y + bu$. In this case, which is common in economics, the last two cases tend to occur:

1. the MHDS has the structure of matrix (7.31) with possibly the coefficients a, b and c linearly dependent upon y and/or q. In this case a steady state (unless at $y = u = 0$) and the MHDS displays unbounded growth. This means that evaluating the MHDS at the steady state the determinant (7.33) will be positive. The existence of a solution depends on the satisfaction of the transversality condition (equation (7.28b)). If this is the case, the solution to the optimal control problem is **non-stationary**, that is, it will be asymptotically unbounded;

2. the MHDS has the structure of matrix (7.31) with $a = \rho$ and $b = 0$. In this case, the $\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) = 0$ and there will be an infinite number of candidate steady states. The system is "anchored" however by the initial condition $y(0) = y_0$, which means that the solution to the optimal control problem exists, it is unique, but it is **stationary**;

Conclusion

Therefore, if there is a unique solution to the infinite-horizon discounted optimal control problem, and if the constitutive functions are monotonic, three types of solutions can occur:

1. a unique time varying solution converging to a steady state;
2. a stationary solution;
3. an time-varying unbounded solution which grows in time such that is satisfies the transversality condition.

A last message: be careful if your optimal control problem has linear constitutive functions.

7.5 Economic applications

We consider the same problems as in the calculus of variations section.

7.5.1 Two simple problems

Example 1: Resource depletion problem

The (non-renewable) resource depletion problem can now be solved by using the Pontriyagin's principle. Recall that the problem is

$$\max_{c(\cdot)} \int_0^\infty e^{-\rho t} \ln(c(t))dt, \ \rho > 0$$

subject to

$$\begin{cases} \dot{w}(t) = -c(t), \ t \in [0, \infty) \\ w(0) = w_0, \text{ given} \\ \lim_{t\to\infty} w(t) \geq 0. \end{cases}$$

In this problem, the control variable is consumption, C, and the state variable is the remaining level of the resource, W. What is the best path for consumption-depletion ?

For applying the Pontryiagin maximum principle we write the current-value Hamiltonian

$$h = \ln(c) - q\,c.$$

The first order conditions are

$$\begin{aligned} c(t) &= 1/q(t) \\ \dot{q} &= \rho q(t) \\ \lim_{t\to\infty} e^{-\rho t} q(t) w(t) &= 0 \\ \dot{w} &= -c(t) \\ w(0) &= w_0 : \end{aligned}$$

and can be written as a planar differential equation in (w, c), together with the initial and the transversality condition is

$$\begin{aligned} \dot{w} &= -c(t) \\ \dot{c} &= -\rho c(t) \\ w(0) &= w_0 \\ \lim_{t\to\infty} e^{-\rho t}\frac{w(t)}{c(t)} &= 0 \end{aligned}$$

If we want to find the solution we must solve the system, together with the conditions on time.

There are several ways to solve it. Here is a simple one. First, define $z(t) \equiv w(t)/c(t)$. Time-differenting and substituting, we get the scalar terminal-value problem

$$\begin{cases} \dot{z} = -1 + \rho z \\ \lim_{t\to\infty} e^{-\rho t} z(t) = 0 \end{cases}$$

which has a constant solution $z(t) = \frac{1}{\rho}$ for every $t \in [0, \infty)$. Second, substitute $c(t) = w(t)/z(t) = \rho w(t)$. therefore,

$$\begin{cases} \dot{w} = -c(t) = -\rho w(t) \\ w(0) = w_0 \end{cases}$$

Then $w^*(t) = w_0 e^{-\rho t}$ for $t \in [0, \infty)$ and $c^*(t) = \rho w^*(t)$.

Characterization of the solution: there is asymptotic extinction

$$\lim_{t\to\infty} w^*(t) = 0,$$

at a speed given by the half-life of the process

$$\tau \equiv \left\{ t : \ w^*(t) = \frac{w(0) - w^*(\infty)}{2} \right\} = -\frac{\ln(1/2)}{\rho}$$

if $\rho = 0.02$ then $\tau \approx 34.6574$ years.

Example 2: the consumption-savings problem

Problem: find the functions $(a(t), c(t))$ pair that maximizes the functional

$$\max_{c(\cdot)} \int_0^\infty e^{-\rho t}\frac{c(t)^{1-\theta} - 1}{1-\theta} dt, \ \rho > 0$$

subject to

$$\begin{cases} \dot{a}(t) = y - c(t) + r\,a, \ t \in [0, \infty) \\ a(0) = a_0, \text{ given} \\ \lim_{t\to\infty} a(t)^{-r\,t} \geq 0. \end{cases}$$

In this problem, the control variable is consumption, c, and the state variable is the level of net wealth, a. The current value Hamiltonian is

$$h(a,c,q) = \frac{c^{1-\theta}-1}{1-\theta} + q\,(y-c+r\,a)$$

and the first order conditions according to the Pontiyagin's principle are

$$\begin{cases} c(t)^{-\theta} = q(t) \\ \dot{q} = q\,(\rho - r) \\ \dot{a} = y - c + r\,a \\ a(0) = a_0 \\ \lim_{t\to\infty} q(t)\,a(t)e^{-\rho t} = 0 \end{cases}$$

As

$$\frac{\dot{q}}{q} = -\theta\,\frac{\dot{c}}{c}$$

we can obtain the solution by solving the mixed initial-terminal value problem for ODE's

$$\begin{cases} \dot{a} = y - c + r\,a \\ \dot{c} = \gamma\,c \\ a(0) = a_0 \\ \lim_{t\to\infty} c(t)^{-\theta} a(t)\,e^{-\rho t} = 0 \end{cases}$$

where again $\gamma \equiv \dfrac{r-\rho}{\theta}$. We present and discuss next the solution to this problem.

7.5.2 Qualitatively specified problems

Next we present a general Ramsey (1928) model in which the behavioral functions are qualitatively specified. This allows us to study the qualitative solution to the optimal control problem.

The Ramsey problem is:

$$\begin{aligned} &\max_{c(\cdot)} \int_0^\infty e^{-\rho t} U(c(t))dt,\ \rho > 0, \\ &\text{subject to} \\ &\dot{k}(t) = F(k(t)) - c(t),\ t \in [0,\infty) \\ &k(0) = k_0 \text{ fixed} \\ &\lim_{t\to\infty} e^{-\rho t} k(t) \geq 0. \end{aligned}$$

We also assume that $(k,c) : \mathbb{R}_+ \to \mathbb{R}^2_+$. In this problem the control variable is c and the state variable is the stock of capital k.

The utility and the production functions, $u(c)$ and $F(k)$, are usually assumed to have the following properties: Increasing, concave and Inada :

$$U'(.) > 0,\ U''(.) < 0, F'(.) > 0,\ F''(.) < 0$$

$$U'(0) = \infty,\ U'(\infty) = 0,\ F'(0) = \infty,\ F'(\infty) = 0.$$

Although we do not have explicit utility and production functions we can still characterize the optimal consumption-accumulation process (we are using the Grobman-Hartmann theorem).

The current-value Hamiltonian is

$$h(c, k, q) = U(c) + q\,(F(k) - c)$$

The necessary (and sufficient) conditions according to Pontriyagin's maximum principle are

$$\begin{aligned} U^{'}(c(t)) &= q(t) \\ \dot{q} &= q(t)\left(\rho - F^{'}(k(t))\right) \\ \lim_{t\to\infty} e^{-\rho t}\, q(t)\, k(t) &= 0 \\ \dot{k} &= F(k(t)) - c(t) \\ k(0) &= k_0 \end{aligned}$$

The MHDS and the initial and transversality conditions become

$$\begin{aligned} \dot{k} &= F(k(t)) - c(t) \\ \dot{c} &= \frac{c(t)}{\theta(c(t))}\left(F^{'}(k(t)) - \rho\right) \\ k(0) &= k_0 > 0 \\ 0 &= \lim_{t\to\infty} e^{-\rho t} U^{'}(c(t)) k(t) \end{aligned}$$

where $\theta(c) = -\frac{U^{''}(c)\,c}{U^{'}(c)} > 0$ is the inverse of the elasticity of intertemporal substitution.

The MHDS has no explicit solution (it is not even explicitly defined) : we can only use **qualitative methods**. They consist in:

- determining the steady state(s): $(\bar{c}, \bar{k})$
- characterizing the linearised dynamics (it is useful to build a phase diagram).

The steady state (if $k > 0$) is

$$\begin{aligned} F^{'}(\bar{k}) &= \rho \Rightarrow \bar{k} = (F^{'})^{-1}(\rho) \\ \bar{c} &= F(\bar{k}) \end{aligned}$$

The linearized MHDS is

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \rho & -1 \\ \frac{\bar{c}}{\theta(\bar{c})}F''(\bar{k}) & 0 \end{pmatrix} \begin{pmatrix} k(t)-\bar{k} \\ c(t)-\bar{c} \end{pmatrix}$$

where we denote $D\mathbf{M}$ the Jacobian matrix. The jacobian J has trace and determinant:

$$\text{tr}(D\mathbf{M}) = \rho,\ \det(D\mathbf{M}) = \frac{\bar{c}}{\theta(\bar{c})}F''(\bar{k}) < 0$$

the steady state $(\bar{c}, \bar{k})$ is a saddle point. The eigenvalues of $D\mathbf{M}$ are

$$\lambda_s = \frac{\rho}{2} - \sqrt{\Delta}\ < 0,\ \ \lambda_u = \frac{\rho}{2} + \sqrt{\delta}\ > \rho > 0$$

where the discriminant is

$$\Delta = \left(\frac{\rho}{2}\right)^2 - \frac{\bar{c}}{\theta(\bar{c})}F''(\bar{k}) > \left(\frac{\rho}{2}\right)^2.$$

and the eigenvector matrix of $D\mathbf{M}$ is

$$\mathbf{P}\ = (\mathbf{P}^s\mathbf{P}^u) = \begin{pmatrix} 1 & 1 \\ \lambda_u & \lambda_s \end{pmatrix}$$

Then the approximate solution for the Ramsey problem, in the neighbourhood of the steady state, is

$$\begin{pmatrix} k^*(t) \\ c^*(t) \end{pmatrix} = \begin{pmatrix} \bar{k} \\ \bar{c} \end{pmatrix} + k_0 \begin{pmatrix} 1 \\ \lambda_u \end{pmatrix} e^{\lambda_s t},\ t \in [0,\infty) \tag{7.34}$$

Then the local stable manifold has slope higher than the isocline $\dot{k}(C,K) = 0$

$$\left.\frac{dc}{dk}\right|_{W^s} = \lambda_u > \left.\frac{dc}{dk}\right|_{\dot{k}} = F'(\bar{k}) = \rho$$

Geometrically (see figure 7.1) the **approximate** solution (7.34) belongs to the stable sub space E^s

$$E^s = \{\ (k,c) : (c-\bar{c}) = \lambda_u(k-\bar{k})\}$$

while the **exact** solution belongs to the stable manifold W^s (which cannot be determined explicitly). Observe that while the slope of the isocline in the neighborhood of the steady is flatter then the slope of the stable manifold

$$\left.\frac{dc}{dk}\right|_{\dot{k}=0} = F'(\bar{k}) = \rho < \left.\frac{dc}{dk}\right|_{W^s} = \lambda_u$$

meaning that the solution approaches the steady state by accumulating (reducing) capital is the initial capital level is smaller (bigger) than the steady state level.

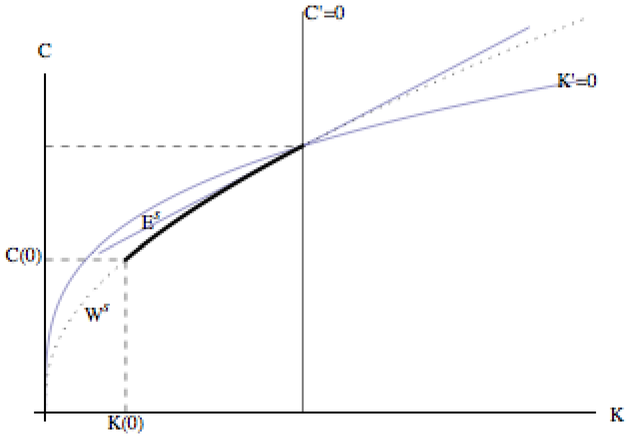

Figure 7.1: The phase diagram for the Ramsey model: it depicts the isoclines $\dot{c}=0$ and $\dot{k}=0$, the stable manifold W^s and the stable eigenspace, E^s, which is tangent asymptotically to the stable manifold. The exact solution follows along the stable manifold, but we have determined just the approximation along the stable eigenspace.

7.5.3 Unbounded solutions

In the previous section we saw that if the solution converges to a steady state we can have a qualitative characterization of the solution appealing to the Grobman-Hartman theorem. However, in some cases, in particular in endogenous growth theory models, solutions may not converge to a steady state, or the solution which interests us can be unbounded in time.

In particular, the consumer-saver problem may have an unbounded solution:

$$\begin{aligned} &\max_{c(\cdot)} \int_0^\infty e^{-\rho t} U(C(t))dt, \ \rho > 0, \\ &\text{subject to} \\ &\dot{A}(t) = Y - C + r\,A, \ t \in [0,\infty) \\ &A(0) = A_0 \text{ fixed} \\ &\lim_{t\to\infty} e^{-\rho t} A(t) \geq 0. \end{aligned}$$

If we write the MHDS in the (A, Q) space, we have

$$\begin{cases} \dot{A} = Y - Q^{-\frac{1}{\theta}} + rA \\ \dot{Q} = Q(\rho - r) \end{cases}$$

the solution of the optimal control problem are the solutions of that MHDS together with the inicial and transversality conditions

$$A(0) = a_0, \ \lim_{t\to\infty} Q(t)A(t)e^{-\rho t} = 0.$$

There are two interesting cases. First, if $r=\rho$ then there is an infinity of stationary solutions satisfying $Q^{-\frac{1}{\theta}}=Y+rA$. Second, if $r\neq\rho$ it has no steady state in $\mathbb{R}$. To see this note that, $\dot{Q}=0$ if and only if $Q=0$ but then $\dot{A}=0$ can only be reached asymptotically when $A\to\infty$.

We can have a clearer characterization if we recast the problem in the (A,C) spac. Recall that in this case we have the MHDS

$$\begin{cases}\dot{A}=Y-C+rA\\ \dot{C}=\gamma C,\end{cases}$$

where

$$\gamma\equiv\frac{r-\rho}{\theta},$$

which, for the moment, we assume has an ambiguous sign.

The solution of the optimal control problem are the solutions of that MHDS together with the inicial and transversality conditions

$$\begin{cases}A(0)=a_0,\\ \lim_{t\to\infty}c(t)^{-\frac{1}{\theta}}A(t)e^{-\rho t}=0.\end{cases}$$

The MHDS is linear planar ODE with coefficient matrix is

$$\mathbf{A}=\begin{pmatrix}r & -1\\ 0 & \gamma\end{pmatrix}$$

that has eigenvalues

$$\lambda_-=\gamma,\ \lambda_+=r>0.$$

and has eigenvector matrix

$$\mathbf{P}=\begin{pmatrix}1 & 1\\ r-\gamma & 0\end{pmatrix}$$

The solution to the MHDS is, for $\gamma\neq 0$

$$\begin{pmatrix}A(t)\\ c(t)\end{pmatrix}=\begin{pmatrix}-\frac{Y}{r}\\ 0\end{pmatrix}+h_-\begin{pmatrix}1\\ r-\gamma\end{pmatrix}e^{\gamma t}+h_+\begin{pmatrix}1\\ 0\end{pmatrix}e^{rt}.$$

For later use, observe that the trajectories starting from $A(0)=a_0$ and travelling along the eigenspace associated to eigenvalue λ_- are

$$\begin{pmatrix}A(t)\\ c(t)\end{pmatrix}=\begin{pmatrix}-\frac{Y}{r}\\ 0\end{pmatrix}+(A_0+\frac{Y}{r})\begin{pmatrix}1\\ r-\gamma\end{pmatrix}e^{\gamma t}.$$

that is

$$\mathbb{E}^-=\left\{\ (A,C)\in\mathbb{R}\times\mathbb{R}_+:\ C=(r-\gamma)\left(A+\frac{Y}{r}\right)\right\}.$$

We saw that the only requirement for the transversality condition to be me, and therefore for the optimal control problem to have a solution was $r>\gamma$. Even if we keep this assumption, three cases are possible

1. if $r < \rho$ then $\lambda_- = \gamma < 0$ and and the steady state $(\bar{A}, \bar{C}) = (-Y/r, 0)$ is a saddle-point. The solution of the optimal control problem, which lies along the stable manifold converges to $c^*(\infty) = 0$ and $A^*(\infty) = -Y/r < 0$. The steady state is a saddle point. The intuition is: the consumer is more impatient than the market and therefore will be asymptotically a debtor to a point in which it can collateralize the debt by its human capital $A(\infty) + H(0) = 0$;
2. if $\gamma < r = \rho$ then $\lambda_- = 0$ and the solution is constant $c^*(t) = Y + rA_0$ and $A^*(t) = A_0$ for all $t \in [0, \infty)$. This was the case corresponding to the existence of an infinite number of equilibria when the characterization is conducted in the (A, Q) space;
3. if $r > \rho$ then $\lambda_- = \gamma > 0$ and the steady state $(\bar{A}, \bar{C}) = (-Y/r, 0)$ is an unstable node. In this case, there are admissible solutions only if $A_0 \geq -Y/r$, otherwise consumption would be negative. However, if $A_0 > -Y/r$ there is an admissible solution to the optimal control problem but it is unbounded.

The question the last case poses is the following. First, if we look at the MHDS as a dynamical system we would say that it is unstable but most of the qualitative theory of ODE characterizes the dynamics close to a steady state. But we already found that this case is indeed a solution to the optimal control problem. How can we reconcile the two points ?

A way to deal with the last type of behavior is to consider convergence of the solution to a kind of invariant structure and to consider convergence to that structure. An approach which is used in the economic growth literature (see Acemoglu (2009)) is to consider convergence to an exponential solution, called **balanced growth path**, such that the initial and the transversality conditions hold.

The method proceeds along five steps.

First, define the variables using multiplicative deviations along an exponential trends with proportional growth rates. In our case we try the case in which the rates of growth are equal

$$A(t) = a(t)e^{gt}, \ c(t) = c(t)e^{gt}$$

Second, obtain the dynamic system for the detrended variables (a, c). If we observe that

$$\frac{\dot{a}}{a} = \frac{\dot{A}}{A} - g, \ \frac{\dot{c}}{c} = \frac{\dot{C}}{C} - g,$$

we get

$$\begin{cases} \dot{a} = Ye^{-gt} - c + (r - g)a \\ \dot{c} = (\gamma - g)c \end{cases}$$

Third, obtain g from a stationary solution to the system in detrended variables. In our case setting $g = \gamma$ transforms the previous system to

$$\begin{cases} \dot{a} = Ye^{-\gamma t} - c + (r - \gamma)a \\ \dot{c} = 0 \end{cases}$$

which implies that $c(t) = \bar{c}$ which is an unknown constant. Setting $a(0) = A_0$ and $c(t) = \bar{c}$ we can solve the equation for the detrended asset holdings

$$a(t) = \left(A_0 - \frac{\bar{c}}{r-\gamma} + \frac{Y}{r}\left(1 - e^{-rt}\right)\right) e^{(r-\gamma)t} + \frac{\bar{c}}{r-\gamma}.$$

Fourth, we can determine $\bar{c}$ from the transversality condition

$$\begin{aligned}
\lim_{t\to\infty} (c(t))^{-\theta} A(t) e^{-\rho t} &= \lim_{t\to\infty} \bar{c}^{-\theta} e^{(\gamma(1-\theta)-\rho)t} a(t) = \\
&= \lim_{t\to\infty} \bar{c}^{-\theta} e^{(\gamma(\theta-1)-\rho+r-\gamma)t} \left(A_0 - \frac{\bar{c}}{r-\gamma} + \frac{Y}{r}\left(1 - e^{-rt}\right) + \frac{\bar{c}}{r-\gamma} e^{-(r-\gamma)t}\right) = \\
&= \lim_{t\to\infty} \bar{c}^{-\theta} \left(A_0 - \frac{\bar{c}}{r-\gamma} + \frac{Y}{r}\left(1 - e^{-rt}\right) + \frac{\bar{c}}{r-\gamma} e^{-(r-\gamma)t}\right) = \\
&= \bar{c}^{-\theta}\left(A_0 - \frac{\bar{c}}{r-\gamma} + \frac{Y}{r}\right) = 0
\end{aligned}$$

if and only if $\bar{c} = c^* = (r-\gamma)\left(A_0 + \frac{Y}{r}\right)$.

At last we get the solution

$$c^*(t) = c^* e^{\gamma t},\ A^*(t) = a^*(t) e^{\gamma t}$$

where

$$c^* = (r-\gamma)\left(a_0 + \frac{Y}{r}\right),\ a^*(t) = A_0 + \frac{Y}{r}\left(1 - e^{-\gamma t}\right).$$

We see that

$$c^*(t) = (r-\gamma)\left(A^*(t) + \frac{Y}{r}\right), \text{for } t \in [0, \infty)$$

which means that the solution to the optimal control problem evolves along the eigenspace associated to the eigenvalue $\lambda\_$ (see figure 7.2).

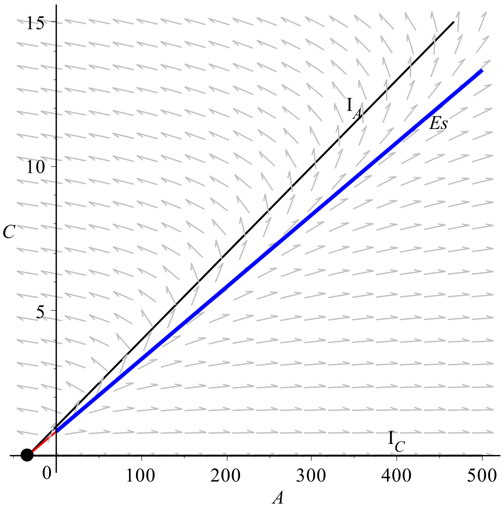

Figure 7.2: Phase diagram for the benchmark consumer problem for the case $r > \gamma$.

If $r < \rho$, and therefore $\gamma < 0$, the solution evolves along the eigenspace associated to $\lambda\_$ butit converges to the steady state in which $A(\infty) = -Y/r$. In this case $\mathbb{E}^- = \mathbb{E}^s$ that is this is the stable eigenspace (which as the model is linear is the stable manifold).

From this we have a geometrical interpretation of the solution to the optimal control problem: if $r \neq \rho$ the solution will belong to the eigenspace $\mathbb{E}^-$, and it converges to the steady state if $r < \rho$ and diverges from it if $r > \rho$.

This illustrates, and reinforces, the fact that interpreting phase diagrams for MHDS of optimal control problems should be done with care: if the optimal control problem has a single solution, the geometrical analog of it is also unique.

7.6 Bibliography

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- Applications to economic growth: Acemoglu (2009).
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- Brief histories of optimal control: Sargeant (2000) or Peschl and Plail (2009)

Chapter 8

Introduction to the dynamic programming approach

The dynamic programming principle provides another approach to solve optimal control problems. It is based upon the principle of dynamic programming that states basically that finding an optimal solution to the problem may be achieved by finding the best rule for the immediate (infinitesimal in continuous time) step. In the literature it is associated to a recursive approach to solving the problem or to finding a feedback rule.

The first order conditions according to the principle of dynamic programming are represented by the Hamilton-Jacobi-Bellman (HJB) equation which is a partial differential equation (for the finite horizon problem) or a implicit ordinary differential equation (for the infinite horizon problem).

It has some advantages and some disadvantages when compared to the maximum principle approach:

- on the plus side: it provides a clearer intuition to dynamic optimization, it allows for an easier extension to stochastic optimal control problems, and, it is preferred by researchers who prefer to solve problems numerically;
- on the minus side: giving place to a differential equation, the terminal condition to solving that equation is not always clear (in fact it is a way to circumvent the need to a terminal condition; studying the differentiability properties of the HJB equation requires advanced functional analysis; and does not lead immediately to the use of the qualitative theory of ODE's to qualitative analysing of the solution. However, w will see that this shortcoming can be eliminated by the use of the envelope theorem.

8.1 The finite horizon case

Consider again the free terminal state optimal control problem: among functions $y \in \mathrm{Y}$ and $u \in U$ satisfying

$$\dot{y} = G(y(t), u(t), t), \text{ for } t \in [0, \bar{t}] \tag{8.1}$$

and $y(0) = y_0$ find the pair (y^*, u^*) that maximize the functional

$$\mathsf{J}[y, u] \equiv \int_0^{\bar{t}} F(t, y(t), u(t)) dt \tag{8.2}$$

where $\bar{t}$ is given and $y^*(\bar{t})$ is free.

Proposition 1 (Necessary conditions according to the principle of dynamic programming). *Consider the optimal state and control functions $y^* \in \mathrm{Y}$ and $u^* \in U$ for the optimal control problem with free terminal state. Then the* ***Hamilton-Jacobi-Bellman*** *equation must hold*

$$-V_t(t, y) = \max_{u \in \mathcal{U}} \left\{ F(t, y, u) + V_y(t, y) G(t, y, u) \right\} \tag{8.3}$$

for all $t \in [0, \bar{t})$ and all $y \in Y \subseteq \mathbb{R}$.

Proof. (heuristic) We define the functional over the state and control functions continuing from an arbitrary time $t \geq 0$: $(y, u) : [t, \bar{t}] \to Y \times U \subseteq \mathbb{R}^2$

$$\mathsf{J}[y, u](t) = \int_t^{\bar{t}} F(s, y(s), u(s)) ds.$$

and call **value function** to

$$V(t, y(t)) \equiv \max_{(u(s) | s \in [t, \bar{t}])} \mathsf{J}[y, u; t]$$

for $y(t) \in Y$.

The **Principle of dynamic programming optimality** states the following: for every $(t, y) \in [0, \bar{t}] \times Y$ and every $\Delta t \in (0, \bar{t} - t]$ the value function satisfies

$$V(t, y(t)) = \max_{(u(s) | s \in [t, t+\Delta t])} \left\{ \int_t^{t+\Delta t} F(s, y(s), u(s)) ds + V(t + \Delta t, y(t + \Delta t)) \right\}$$

where

$$y(t + \Delta t) = y(t) + G(t, y(t), u(t)) \Delta t + o(\Delta t).$$

Performing a first-order Taylor expansion we get

$$V(t + \Delta t, y(t + \Delta t)) = V(t, y(t)) + V_t(t, y(t)) \Delta t + V_y(t, y(t)) G(t, y(t), u(t)) \Delta t + o(\Delta t)$$

(this requires that V is C^1). If the interval Δt is sufficiently small we can use the mean-value theorem

$$\int_t^{t+\Delta t} F(s, y(s), u(s)) ds = F(t, y(t), u(t)) \Delta t$$

Then

$$V(t,y(t)) = \max_{(u(s)|s\in[t,t+\Delta t])} \left\{ F(t,y(t),u(t))\Delta t + V(t,y(t)) + V_t(t,y(t))\Delta t + V_y(t,y)g(t,y(t),u(t))\Delta t + o(\Delta t) \right\}.$$

Cancelling out $V(t,y(t))$, dividing by Δt, taking $\Delta t \to 0$ and observing that the pair $(t,y(t))$ is an arbitrary element of $T \times Y$ we get the HJB equation (8.3). □

For solving the optimal control problem, while the Pontriagyin's principle provides necessary conditions in a form of a initial-terminal value problem for a planar ODE, the principle of the dynamic programming provides a formula for evaluating the value of our resource in a recursive way and independent of time.

The HJB equation (8.3) is a PDE (partial differential equation).

8.2 Infinite horizon discounted optimal control problem

The infinite horizon discounted optimal control problem is, again, to find functions $u^* \in \mathrm{U}$ and $u^* \in \mathrm{Y}$ satisfying

$$\begin{cases} \dot{y} = g(y(t),u(t)), & t \in [0,\infty) \\ y(0) = y_0, & \\ \lim_{t\to\infty} h(t)y(t) \geq 0 & \end{cases}$$

that maximize the objective functional

$$\mathsf{J}[y,u] \equiv \int_0^\infty e^{-\rho t} f(y(t),u(t))dt$$

Proposition 2 (Necessary conditions according to the principle of dynamic programming for the infinite horizon problem). *Let (y^*,u^*) be the solution to the discounted infinite horizon problem. Then it satisfies the HJB equation*

$$\rho v(y) = \max_u \{ f(y,u) + v'(y)g(y,u)\} \tag{8.4}$$

Proof. For $y(t) = y$ the value function is

$$V(t,y) \equiv \int_t^\infty e^{-\rho s} f(y^*(s),u^*(s))ds$$

Multiplying by a inverse of the discount factor, the value function becomes independent of the initial time,

$$e^{\rho t}V(t,y) = \int_t^\infty e^{-\rho(s-t)} f(y^*(s),u^*(s))ds = v(y).$$

If we take derivatives of $V(t,y) = e^{-\rho t}v(y)$, we have $V_t(t,y) = -\rho\, e^{-\rho t}v(y)$ and $V_y(t,y) = e^{-\rho t}v'(y)$, which after substituting in equation (8.3) yields equation (8.4). □

In the case of the discounted infinite horizon the HJB equation is not a PDE but an ODE in implicit form. In order to see this we need to determine another important element of the DP approach: the policy function.

If we define the function $h(u, y) \equiv f(y, u) + v^{'}(y)g(y, u)$ the HJB equation (8.4) can be written as

$$\rho v(y) = \max_{u} h(u, v).$$

We can obtain the optimal control from the first-order condition

$$\frac{\partial h(u, y)}{\partial u} = 0.$$

If function $h(u, v)$ is monotonic as regards u, by appealing to the implicit function theorem, we can obtain the optimal control as a function of the state variable, $u^* = \pi(y)$. Function $\pi(.)$ in the DP literature is called **policy function**. It gives the optimal control as a function of the state variable. This is why it is called a **feedback control** problem.

The reason for this is the following. If we substitute the policy function in equation (8.4) we finally obtain the HJB equation as an ODE in implicit form

$$\rho v(y) = f(\pi(y), y) + v^{'}(y)g(\pi(y), y)$$

where the state variable y is the independent variable and the value function, $v(y)$, is the unknown function.

If we are able to determine a solution to this equation, we can usually specify the utility function, which means that we are able to obtain the optimal control as a function of the state variable. We can obtain the solution to the optimal control problem by substituting in the ODE constraint to get

$$\dot{y} \ = g(y, \pi(y)), \ t \in [0, \infty)$$

which, together with the initial condition $y(0) = y_0$, would, hopefully, allow for the determination of the solution for the state variable.

If we can find the policy function, then obtaining the optimal dynamics for y reduces to solving an initial-value problem instead of a mixed initial-terminal value problem (or two-point boundary value problem) as is the case when we use the calculus of variations of the Pontriyagin's principle approaches.

However, only in a very small number of cases we can obtain closed form solutions to the HJB equation. Next we show some cases in which this is possible.

8.3 Applications

8.3.1 Example 1: The resource depletion problem

We solve again resource-depletion problem for an infinite horizon

$$\max_{C} \int_{0}^{\infty} e^{-\rho t} \ln(C(t))dt, \text{ s.t } \dot{W} = -C, \ W(0) = W_0$$

by using the DP principle.

The HJB equation is

$$\rho v(W) = \max_{C} \Big[\ \ln(C) + v^{'}(W)(-C) \Big]$$

Policy function

$$\frac{1}{C^{*}} - v^{'}(W) = 0 \Leftrightarrow C^{*} = \left(v^{'}(W)\right)^{-1}$$

Then the HJB becomes

$$\rho v(W) = - \ \ln(v^{'}(W)) - 1$$

The textbook method for solving the HJB equation through is by using the **method of undetermined coefficients** after we make a conjecture over the form of the value function (no constructive way here).

Assume the trial function

$$v(W) = a + b\ln(W)$$

As $v^{'}(W) = b/W$ and substituting and collecting terms we get

$$\rho a + 1 + \ln(b) = \ln(W)\,(1 - \rho b)$$

then $b = 1/\rho$ and $a = (\ln \rho - 1)/\rho$.

Then:

$$v(W) = \frac{\ln \rho - 1 + \ln(W)}{\rho}, \ C^{*} = (v^{'}(W))^{-1} = \rho W$$

A second method: the HJB equation is an ODE, where W is the independent variable, so we can try to solve it (this is a constructive method).

The HJB is equivalent to

$$v^{'}(W) = e^{-(1+\rho v(W))}$$

ODE $y^{'}(x) = e^{(a+by(x))}$ has the closed form solution

$$y(x) = \frac{1}{b} \left(-a + \ln\left(-\frac{1}{b(k+x)} \right)\right)$$

where k is an arbitrary constant. Then we determine

$$v(W) = -\frac{1}{\rho}\left(1 + \ln\left(\frac{1}{\rho(W+k)}\right)\right)$$

and

$$C^{*} = \left(V^{'}(W)\right)^{-1} = \rho(W+k)$$

Substituting in the constraint $\dot{W} = -C = -\rho(W+k)$, we get the solution

$$W(t) = -k + (W(0)+k)e^{-\rho t}.$$

The problem is somewhat incompletely specified, which reveals a potential problem when using the DP approach.

In our case, as it is natural to assume that $\lim_{t\to\infty} W(t) = 0$ we would obtain $k = 0$ and therefore we would get the same solution as from using the CV and Pontriyagin's approaches:

$$C^*(t) = \rho W_0 e^{-\rho t},\ W^*(t) = W_0 e^{-\rho t},\ \text{for } t \in [0,\infty).$$

8.3.2 Example 2: The benchmark consumption-savings problem

Applying the HJB equation (8.4) to our problem we have

$$\rho v(A) = \max_C \left\{ \frac{C^{1-\theta}-1}{1-\theta} + v^{'}(A)(Y - C + rA) \right\}. \tag{8.5}$$

Define a indirect utility function by

$$\tilde{u}(v'(A)) = \max_C \left\{ \frac{C^{1-\theta}-1}{1-\theta} - v^{'}(A)\, C \right\}$$

and total wealth, summing up human and financial wealth, by $W(A) \equiv \dfrac{Y}{r} + A$, then the HJB equation (8.5) at the optimum is a implicit ODE

$$\rho v(A) = \tilde{u}(v'(A)) + r\, v^{'}(A)\, W(A). \tag{8.6}$$

Solving the static utility problem we get the optimum policy for consumption

$$C^* = \pi(A) \equiv \left(v^{'}(A)\right)^{-\frac{1}{\theta}}.$$

as a function of the (unknown) marginal value function, and upon substitution yields

$$\tilde{u}(v'(A)) = \frac{1}{1-\theta}\Big(\left(v^{'}(A)\right)^{\frac{\theta-1}{\theta}} - 1 \Big).$$

Therefore, equation (8.6) becomes

$$\rho\, v(A) = \frac{\theta}{1-\theta}\left(v^{'}(A)\right)^{\frac{\theta-1}{\theta}} - \frac{1}{1-\theta} + r\, v^{'}(A)\, W(A) \tag{8.7}$$

To solve this (implicit ODE) equation, we use again the method of undetermined coefficients. Conjecturing the trial function

$$v(A) = a + b\, W(A)^{1-\theta},$$

with arbitrary parameters a and b. Then

$$v^{'}(A) = b\,(1-\theta)\, W(A)^{-\theta}$$

and after substitution in equation (8.7) we get

$$a\rho + \frac{1}{1-\theta} = W(A)^{1-\theta}\, b\, \theta \Big[\Big(b\,(1-\theta) \Big)^{-1/\theta} - (r-\gamma) \Big]$$

where we have again $\gamma \equiv (r-\rho)/\sigma$. Setting both sides to zero, yields

$$a = \frac{1}{\rho(\theta-1)} \text{ and } b = \frac{(r-\gamma)^{-\theta}}{1-\theta}$$

Then, the value function is

$$v(A) = \frac{1}{1-\theta}\Big[(r-\gamma)^{-\theta} \left(\frac{Y}{r} + A \right)^{1-\theta} - \frac{1}{\rho} \Big].$$

Taking the derivative as regards A and substituting in the policy function for C, we find the optimal consumption in feedback form

$$C^*(A) = (r-\gamma)\left(\frac{Y}{r} + A\right)$$

which only makes sense if $r > \gamma$.

We can get the optimal asset path by substituting optimal consumption in the budget constraint

$$\dot{A}^* = Y + rA - C^*(A) = \gamma \left(\frac{Y}{r} + A\right).$$

Solving this equation with $A(0) = A_0$ we get the optimal paths for asset holdings

$$A^*(t) = -\frac{Y}{r} + \left(\frac{Y}{r} + A_0\right) e^{\gamma t}, \text{ for } t \in [0, \infty),$$

and consumption

$$C^*(t) = (r-\gamma)\left(\frac{Y}{r} + A_0\right) e^{\gamma t}, \text{ for } t \in [0, \infty).$$

Exercise Prove, by setting $\theta = 1$, that the value function for $u(C) = \ln(C)$ is

$$V(A) = \frac{1}{\rho}\Big[\frac{r-\rho}{\rho} + \ln(\rho\, W(A)) \Big].$$

Hint: use the property $f(x) = \exp(\ln f(x))$ and use the l'Hôpital theorem.

The utility function is a generalized logarithm $u(C) = \ln_\theta(C) = \dfrac{C^{1-\theta} - 1}{1-\theta}$. Sometimes in the literature people write

$$u(C) = \begin{cases} \dfrac{C^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1 \\ \ln(C) & \textit{if}\, \theta = 1 \end{cases}$$

The problem with this formulation is that if we cannot obtain the value function for the logarithm utility by setting the limit of $\theta = 1$ for the general case $\theta = 1$, which is

$$v(A) = \frac{(r-\gamma)^{-\theta}}{1-\theta} W(A)^{1-\theta}.$$

8.3.3 Example 3: The Ramsey model

The HJB for the Ramsey model is

$$\rho v(k) = \max_{c} \left\{ u(c) + v^{'}(k)\left(F(k) - c\right) \right\}$$

The optimality condition is

$$u'(c) = v'(k)$$

if u is sufficiently smooth then we obtain the policy function $c = C(k) = (u')^{-1}(v'(k))$. Substituting back in the HJB equation yields the implicit ODE in $v(k)$

$$\rho v(k) = u(C(k)) + v^{'}(k)(F(k) - C(k))$$

which does not have a closed form solution in general.

Exercise: for the case in which $u(c) = \dfrac{c^{1-\theta} - 1}{1-\theta}$ and $F(k) = k^{\alpha}$, such that $\theta = \alpha$ prove that a closed form solution can be found.

8.3.4 Example 4: The AK model

The Rebelo (1991) AK model can be seen as a special case of the previous problem in which the HJB function is

$$\rho v(K) = \max_{C} \left\{ \frac{C^{1-\theta}}{1-\theta} + v^{'}(K)\left(AK - C\right) \right\}$$

Using the same steps as before, we get

$$\rho v(K) = \frac{\theta}{1-\theta}\left(v^{'}(K)\right)^{\frac{\theta-1}{\theta}} + v^{'}(K)AK \tag{8.8}$$

To solve the equation we use again the method of undetermined coefficients and find

$$v(K) = \frac{\left((A-\gamma)K\right)^{1-\theta}}{1-\theta}.$$

where

$$\gamma = \frac{A-\rho}{\theta}.$$

The consumption, in the feedback form is,

$$C^{*}(K) = (A-\gamma)K$$

and the budget constraint of the economy is

$$\dot{K}^{*} = AK^{*} - C^{*}(K) = \gamma K^{*}.$$

Considering the given initial level for capital $K(0) = K_0$ we get the optimal paths for capital and output

$$K^{*}(t) = K_0 e^{\gamma t},\ Y^{*}(t) = AK_0 e^{\gamma t},\ \text{for } t \in [0, \infty).$$

8.4 Relationship with the PMP

Consider the HJB equation (8.4).

The optimal policy function $u^* \; = U(y)$ is obtained from the optimality condition

$$f_u(y, u^*) + v'(y)\, g_u(y, u^*) = 0$$

Write $q = v'(y)$. Next we will show that with this change of variables we will obtain the optimality conditions according to the Pontriyagin's maximum principle.[1]

First, observe that, defining $h(y, u) = f(y, u) + v'(y)\, g(y, u) = f(y, u) + q\, g(y, u)$ at the optimum we have $h_u(y, u^*) = f_u(y, u^*) + q\, g_u(y, u^*) = 0$ is the static optimality condition according to the Pontriyagin's maximum principle.

Taking the derivative of the (8.4), at the optimum, yields

$$\begin{aligned}\rho\, v'(y) &= f_y(y, u^*) + f_u(y, u^*)\, U'(y) + v''(y)\, g(y, u^*) + v'(y) \left(g_y(y, u^*) + g_u(y, u^* \;)\, U'(y)\right) \\ &= f_y(y, u^*) + v''(y)\, g(y, u^*) + v'(y)\, g_y(y, u^*)\end{aligned}$$

using the optimality condition. Therefore, if function $v(\cdot)$ is smooth

$$g(y, u^*) = \frac{\rho\, v'(y) - f_y(y, u^*) - v'(y)\, g_y(y, u^*)}{v''(y)} \; = \frac{\rho\, q - h_y(y, u^*}{v''(y)} \tag{8.9}$$

using the previous definition.The constraint to the problem evaluated at the optimum, $\frac{dy}{dt} = g(y, u^*)$. Taking the time detivative of $q = v'(y)$ implies

$$\frac{dq}{dt} = v''(y)\;\frac{dy}{dt} = v''(y) g(y, u^*) =$$

If we substitute equation (8.9) yields

$$\frac{dq}{dt} = \rho\, q - h_y(y, u^*)$$

which is the multiplier equation from the Pontriyagin's maximum principle [2]

This allows for a qualitative dynamics analiysis of the solution to the optimal control problem obtained via the HJB equation.

8.4.1 Application to the Ramsey model

Consider the Ramsey problem with a CRRA utility function and a Cobb-Douglas production function. The HJB equation is

$$\rho\, v(k) = \max_c \left\{ \frac{c^{1-\theta} - 1}{1-\theta} + v'(k) \left(k^{\alpha} \; - c\right)\right\}$$

---

[1] Although the DP approach is silent to what to do with the transversality condition. However, Ekeland (2010) shows that the solution to the problem exists if there is a steady state vale for the state variable and at the steady state $\bar{y}$ we should have $\rho\, v(\bar{y}) = v'(\bar{y})$.

[2] See (Beckmann, 1968, p.33).

where $\theta > 0$ and $0 < \alpha < 1$. The policy function is $c^* = C(k) = \left(v'(k)\right)^{-\frac{1}{\theta}}$. Then

$$\frac{dc}{dt} = -\frac{c^*\, v''(k)}{\theta\, v'(k)}\, \frac{dk}{dt}.$$

From the envelop theorem, we obtain

$$\rho\, v'(k) = v''(k)\left(k^{\alpha}\ \ - c^*\right) + \alpha\, k^{\alpha-1}\ \ v'(k)$$

as

$$\frac{dk}{dt} = k^{\alpha}\ \ - c^* = \frac{v'(k)\,(\rho - \alpha\, k^{\alpha-1}}{v''(k)}$$

then

$$\frac{dc}{dt} = c\frac{(\alpha\, k^{\alpha-1} - \rho)}{\theta}$$

which is the Ramsey-Keynes equation associated to the PMP.

8.5 Bibliography

- The seminal contribution: Bellman (1957).
- Other references: Beckmann (1968)
- Recent textbook: **?**

Chapter 9

Optimal control of ODE's: extensions

9.1 Introduction

In this chapter we consider some extensions of the simple optimal control problems we dealt in the last chapter

9.2 Singular optimal control

Assume that the Hamiltonian is linear in the control variable. This is equivalent to stating that $H_u\ (y, u, t) = h(y, t)$ and $H_{uu}\ (y, u) = 0$ for all $u \in \mathcal{U}$.

Consider the problem:

$$
\begin{aligned}
\max_{u(\cdot)} \ \mathsf{J}[y,u] \ &= \int_0^T \ F(t, y(t), u(t))\, dt \\
&\text{subject to} \\
&\dot{y} \ = G(t, y(t), u(t)) \\
&y(0) = y_0 \,\text{given} \\
&R(y(T), T) \geq 0.
\end{aligned}
\tag{9.1}
$$

In this case define the Hamiltonian function as

$$H(y, u, \lambda_0, \lambda, t) = \lambda_0\, F(y, u, t) + \lambda\, G(y, u, t).$$

An informal version of the Pontriyagin's principle states the following: If (y^*, u^*) is an optimum, then there is a scalar $\lambda_0 \in \{0, 1\}$ and a piecewise continuous function $\lambda : \mathrm{T} \to R$ such that:

1. for every $t \in [0, T)$

$$
\begin{aligned}
H_u^*(t) &= 0 \\
\dot{\lambda} \ &= -H_y^*(t) \\
\dot{y} \ &= H_\lambda^*\ (t) = G(t, y^*(t), u^*(t))
\end{aligned}
$$

2. at the terminal time $t = T$
$$\lambda(T)\, R^*_y(y(T), T) = 0$$

This problem may require enlarging the domain space, also for the state variable y to the set of piecewise-continuous functions $y : \mathrm{T} \to \mathbb{R}$.

Here we are mostly concerned with the household problem with a linear utility function, that is a isoelastic utility function with an infinitely-valued elasticity of substitution. See (Grass et al., 2008, ch. 3.5) for a complete reference.

Example The household problem with a linear utility function

$$\begin{aligned}\max_{c(\cdot)} \int_0^\infty &\; c(t)\, e^{-\rho\, t}\, dt \\ &\text{subject to} \\ &\dot{a} \;= r\, a - c \\ &a(0) = a_0, \text{ fixed} \\ &\lim_{t\to\infty} \; e^{-r\, t}\; a(t) \geq 0\end{aligned}$$

The Hamiltonian function is

$$H(a, c, \lambda_0, \lambda) = \lambda_0 c + \lambda\, (r\, a - c)$$

where λ_0 is a number and λ is a function of time. The first order conditions (observe that the Hamiltonian function is a concave, although not strictly concave function of (c, a)) are

$$\begin{aligned}\lambda_0 &= \lambda(t),\; t \in \mathrm{T} \\ \dot{\lambda} \;&= \lambda\, (\rho - r),\; t \in \mathrm{T} \\ \lim_{t\to\infty} \; \lambda(t)\, a(t) e^{-\rho t} \;&= 0 \\ \dot{a} \;&= r\, a - c \\ a(0) &= a_0.\end{aligned}$$

Setting $\lambda_0 = 1$ then $\lambda(t) = 1$ for every $t \in [0, \infty)$, which implies $\dot{\lambda} \; = 0$. A solution only exists if $r = \rho$, which we assume to be the case from now on. Solving the budget constraint and substituting in the transversality condition we should have $a_0 = \int_0^\infty \; e^{-\rho\, s} \; c(s)\, ds$. As there are no more constraints on the functional form for $c(t)$, as in the case with constant elasticity of substitution, the solution for c is indeterminate. That is, there is an infinite number of consumption trajectories that satisfies that constraint. In particular, a constant consumption path $c(t) = c^* = \rho\, a_0$ is a solution.

Setting $\lambda_0 = 0$ then $\lambda(t) = 0$ for every $t \in [0, \infty)$, which implies $\dot{\lambda} \; = 0$. However, this does not require that an existence condition is $r = \rho$. However, even if we assume that $r - \rho$ can have any sign, the transversality condition will be satisfied for any trajectories of a and c. Again, the solution is indeterminate.

From the economic point of view the problem is misspecified. In order to overcome this, either we introduce more curvature in the utility function, or we introduce some adjustment costs in consumption, or bounds in the net asset position of the consumer.

9.3 Two-stage optimal control problems

Assume that the independent variable is time $t \in \mathrm{T} = [t_0, t_2]$ but there is a discontinuity in the objective function andor in the constraint to the problem such that

$$F(t) = F(t, y(t), u(t)) = \begin{cases} F_1(t, y(t), u(t)) & \text{if } t_0 \geq t < t_1 \\ F_2(t, y(t), u(t)) & \text{if } t_1 \geq t \leq t_2 \end{cases}$$

and/or

$$G(t) = G(t, y(t), u(t)) = \begin{cases} G_1(t, y(t), u(t)) & \text{if } t_0 \geq t < t_1 \\ G_2(t, y(t), u(t)) & \text{if } t_1 \geq t \leq t_2 \end{cases}$$

where the *switching time* $t_1 \in \mathrm{T} = [t_0, t_2]$, that it, it satisfies $t_0 < t_1 < t_2$, and may be known or may be a decision variable. The optimal control problem in which the switching time is a decision variable is called in the literature a **two-phase optimal control problem**.

The common structure of the problems we address in this section is

$$\begin{aligned} &\max_{u(\cdot)} \mathsf{J}[u, y] = \int_{t_0}^{t_1} F_1(t, y(t), u(t))\, dt + \int_{t_1}^{t_2} F_2(t, y(t), u(t))\, dt \\ &\quad \text{subject to} \\ &\quad \dot{y} = \begin{cases} F_1(t, y(t), u(t)) & \text{if } t_0 \geq t < t_1 \\ F_2(t, y(t), u(t)) & \text{if } t_1 \geq t \leq t_2 \end{cases} \\ &\quad t_0, \text{ and } t_2 \text{ fixed} \\ &\quad y(t_0) = y_0 \text{ fixed} \\ &\quad y(t_2) \geq y_2 \text{ constrained} \end{aligned} \tag{PTS}$$

We consider the following versions of the problem depending on the switching conditions:

$$t_1 \text{ fixed and } y(t_1) \text{ free} \tag{PTS1}$$

$$t_1 \text{ free but subject to } t_0 \leq t_1 \leq t_2 \text{ and } y(t_1) \text{ free} \tag{PTS2}$$

The Hamiltonian function becomes a piecewise continuous, or piecewise differentiable function

$$H(t) = H(t, y(t), u(t), \lambda(t)) = \begin{cases} H_1(t) = F_1(t) + \lambda(t)\, G_1(t) & \text{if } t_0 \geq t < t_1 \\ H_2(t) = F_2(t) + \lambda(t)\, G_2(t) & \text{if } t_1 \geq t \leq t_2 \end{cases}$$

where the co-state variable is $\lambda : \mathrm{T} \to \mathbb{R}$, and $H_i(t) = H_i(t, y(t), u(t), \lambda(t))$ for $i = 1, 2$.

Proposition 1. ***[First order necessary conditions for the two-stage optimal control problem]*** *Let (y^*, u^*) be a solution to the OC problem PTS in which one of the conditions (PTS1), or (PTS2) is introduced. Then there is a piecewise continuous function $\lambda : \mathrm{T} \to \mathbb{R}$, called co-state variable, such that (y^*, u^*, λ) satisfy the following conditions:*

- *the optimality condition:*

$$\begin{aligned} H^*_{1,u}(t) = 0,\ &for\ \ t \in [t_0, t^*_1) \\ H^*_{2,u}(t) = 0,\ &for\ \ t \in [t^*_1, t_2] \end{aligned} \tag{9.2}$$

  where $H^*_{i,u}(t) = H_{i,u}(t, y^*(t), u^*(t), \lambda(t))$ for $i = 1, 2$;

- *the multiplier equation*

$$\begin{aligned} \dot{\lambda}\ \ + H^*_{1,y}(t) = 0,\ &for\ \ t \in [t_0, t^*_1) \\ \dot{\lambda}\ \ + H^*_{2,y}(t) = 0,\ &for\ \ t \in [t^*_1, t_2] \end{aligned} \tag{9.3}$$

  where $H^*_{i,y}(t) = H_{i,y}(t, y^*(t), u^*(t), \lambda(t))$ for $i = 1, 2$;

- *the constraint of the problem:*

$$\begin{aligned} \dot{y}\ \ =&G^*_1(t),\ for\ \ t \in [t_0, t^*_1) \\ \dot{y}\ \ =&G^*_2(t),\ for\ \ t \in [t^*_1, t_2] \end{aligned} \tag{9.4}$$

  where $G^*_i(t) = G_i(t, y^*(t), u^*(t))$ for $i = 1, 2$;

- *the adjoint condition associated to the initial values $(t_0, y(t_0))$: $y^*(t_0) = y_0$;*

- *the adjoint conditions associated to the terminal values $(t_2, y(t_2))$:*

$$\lambda(t_2)\,(y^*(t_2) - y_2) = 0,\ \ and\ \lambda(t_2) \geq 0; \tag{9.5}$$

- *and the adjoint conditions associated to switching conditions (PTS1) and (PTS2) are*
  - *for problem (PTS1): assuming that $t_0 < t_1 < t_2$ the switching condition is*

$$\lambda(t_1^-) = \lambda(t_1^+) \tag{9.6}$$

    where $\lambda(t_1^-) = \lim_{t \uparrow t_1} \lambda(t)$ and $\lambda(t_1^+) = \lim_{t \downarrow t_1} \lambda(t)$ are the limits of the co-state variables determined from the first stage (i.e., from $t \in [t_0, t_1]$) and from the second stage (i.e., from $t \in [t_1, t_2]$) respectively;
  - *for problem (PTS2) there are two conditions,*

$$\lambda(t_1^{*-}) = \lambda(t_1^{*+}) \tag{9.7}$$

where $\lambda(t_1^{*-}) = \lim_{t\uparrow t_1^*} \lambda(t)$ *and* $\lambda(t_1^{*+}) = \lim_{t\downarrow t_1^*} \lambda(t)$ *together with one of the one of the following conditions allowing for the determination of the optimal switching time* t_1^**:*

$$\begin{aligned} t_1^* = t_0 < t_2 &\iff H_1^*(t_1^{*-}) < \ H_2^*(t_1^{*+}) \\ t_0 < t_1^* < t_2 &\iff H_1^*(t_1^{*-}) = \ H_2^*(t_1^{*+}) \\ t_0 < t_1^* = t_2 &\iff H_1^*(t_1^{*-}) > \ H_2^*(t_1^{*+}) \end{aligned} \tag{9.8}$$

where $H_1^*(t_1^{*-}) = \lim_{t\uparrow t_1^*} H_1(t)$ *and* $*_2(t_1^{*+}) = \lim_{t\downarrow t_1^*} H_2^*(t)$*.*

Proof. (Heuristic) Let u^* and y^* be the optimal control and state variable. The value functional, at the optimum, is, for (PTS2) problem,

$$\mathsf{J}^* \ = \int_{t_0}^{t_1^*} F_1^*(t)\,dt + \int_{t_1^*}^{t_2} \ F_2^*(t)\,dt$$

and the associated Lagrangean is

$$\mathsf{L}^* = \mathsf{J}^* + \mu_0\,(t_1^* - t_0) + \mu_2\,(t_2 - t_1^*) + \psi\,(y^*\ (t_2) - y_2),$$

where the complementary slackness conditions should hold

$$\begin{aligned} \mu_0\,(t_1^* - t_0) = 0, \mu_0 \geq 0, &\text{ and } \ t_1^* \geq t_0 \\ \mu_2\,(t_2 - t_1^*) = 0, \mu_2 \geq 0, &\text{ and } \ t_1^* \leq t_2 \\ \psi\,(y^*\ (t_2) - y_2) = 0, \psi \geq 0, &\text{ and } \ y^*(t_2) \leq y_2. \end{aligned}$$

Introducing the admissible perturbations $y(t) = y^*(t) + \varepsilon\,\eta_y(t)$ to the state variable, $u(t) = u^*(t) + \varepsilon\,\eta_u(t)$ to the control variable and $t_1 = t^* \ + \varepsilon\,\tau_1$, yields the Gâteaux differential

$$\begin{aligned} \delta_{\eta(\cdot)}\ \mathsf{L}\ [y^*, u^*; t_1^*] \ &= \int_{t_0}^{t_1^*} \Big[\ H_{1,u}^*(t)\,\eta_u(t) + \Big(H_{1,y}^*(t) + \dot{\lambda}\ (t)\Big)\,\eta_y(t)\Big]\,dt + \\ &+ \int_{t_1^*}^{t_2} \ \Big[\ H_{2,u}^*(t)\,\eta_u(t) + \Big(H_{2,y}^*(t) + \dot{\lambda}\ (t)\Big)\,\eta_y(t)\Big]\,dt \\ &+ \lambda(t_0)\,\eta(t_0) + (\lambda(t_1^{*+}) - \lambda(t_1^{*-}))\,\eta(t_1^*) + (\psi - \lambda(t_2)\,\eta(t_2) \\ &+ \Big(\mu_0 - \mu_2 + H_1^*(t_1^{*-}) - H_2^*(t_1^{*+})\Big)\,\tau_1. \end{aligned}$$

At the optimum $\delta_{\eta(\cdot)}\ \mathsf{L}\ [y^*, u^*; t_1^*] = 0$ should be sarisfied, together with the complementary slackness conditions for admissible perturbations. As the only constraint on the perturbation is $\eta(t_0) = 0$, at the optimum satisfies $H_{1,u}^*(t) = H_{1,y}^*(t) + \dot{\lambda}(t) = \dot{y}\ - G_1^*\ (t) = 0$ for every $t \in [t_0, t_1^{*-})$, $H_{2,u}^*(t) = H_{2,y}^*(t) + \dot{\lambda}(t) = \dot{y}\ - G_2^*\ (t) = 0$ for every $t \in (t_1^{*+}, t?2]$, $\lambda(t_2)(y^*(t_2) - y_2) = 0$, and $\lambda(t_1) = \lambda(t_1)$ if t_1 is fixed, or $\lambda(t_1^{*+}) = \lambda(t_1^{*-})$ together with $\mu_0 - \mu_2 + H_1^*(t_1^{*-}) - H_2^*(t_1^{*+}) = 0$ together with the associated complementary slackness conditions if t_1 should be optimally determined. ☐

Switching costs: there are version of the problem in which there is a switching cost at time $t = t_1$, depending on the state of the problem, as $\Phi(t_1, y(t_1)$. In this case we have

- for the fixed switching time problem (**??**) instead of condition (9.6) we have

$$\lambda(t_1^-) + \Phi_y\ (t_1, y^*(t_1)) = \lambda(t_1^+)$$

$\Phi_y\ (t, y) = \frac{\partial \Phi(t,y)}{\partial y}$.

- for the free switching time problem (**??**) instead of condition (9.7) we have

$$\lambda(t_1^{*-}) + \Phi_y\ (t_1, y^*(t_1^*) = \lambda(t_1^{*+})$$

and instead of conditions (9.8) we have

$$\begin{aligned} t_1^* = t_0 < t_2 &\iff H_1^*(t_1^{*-}) - \Phi_t\ (t_1, y^*(t_1^*) < \ H_2^*(t_1^{*+}) \\ t_0 < t_1^* < t_2 &\iff H_1^*(t_1^{*-}) - \Phi_t\ (t_1, y^*(t_1^*) = \ H_2^*(t_1^{*+}) \\ t_0 < t_1^* = t_2 &\iff H_1^*(t_1^{*-}) - \Phi_t\ (t_1, y^*(t_1^*) > \ H_2^*(t_1^{*+}) \end{aligned}$$

where $\Phi_t\ (t, y) = \frac{\partial \Phi(t,y)}{\partial t}$.

Observations: we can extend this approach to multple switcing times.

References: Tomiyama (1985), Rossana (1989), Makris (2001). For an application to endogenous growth see Boucekkine et al. (2004).

9.4 Constraints on state and control variables

Assume that the independent variable is time $t \in \mathrm{T} = [0, T]$ and one of the following constraints exist: constraints on state variables such that $Q_2(y(t), t) \le 0$, or constraints in the state and/or control variables such that $Q_1(ty(t), u(t), t) \le 0$, for every $t \in \mathrm{T}$.

$$\begin{aligned} \max_{u(\cdot)}\ \mathsf{J}[y, u]\ &= \int_0^T\ F(t, y(t), u(t))\, dt \\ &\text{subject to} \\ &\dot{y}\ = G(t, y(t), u(t)) \\ &y(0) = y_0 \,\text{given} \\ &Q_1(t, y(t), u(t)) \ge 0,\ \text{for any}\ \ t \in \mathrm{T} \\ &Q_2(t, y(t)) \ge 0,\ \text{for any}\ \ t \in \mathrm{T} \\ &R(y(T), T) \ge 0. \end{aligned} \tag{9.9}$$

where $Q_1(\cdot) \ge 0$ is a joint constraint on the state variable and the control variable and $Q_2(\cdot) \ge 0$ is a pure state constraint.

In this problem we have potencial points of discontinuity for the adjoint variable $\lambda(\cdot)$ when the constraints are

In this case an (informal) version of the Pontriyagin maximum principle is presented in Hartl et al. (1995). Define the Hamiltonian function

$$H(y,u,\lambda_0,\lambda,t) = \lambda_0\,F(y,u,t) + \lambda\,G(y,u,t),$$

and the Lagrangean

$$L(y,u,\lambda_0,\lambda,t) = H(y,u,\lambda_0,\lambda,t) + \nu_1\,Q_1(y,u,t) + \nu_2\,Q_1(y,t).$$

An informal version of the Pontriyagin's principle states the following: If (y^*,u^*) is an optimum, then there is a scalar $\lambda_0 \in \mathbb{R}$, a piecewise function $\lambda : \mathrm{T} \to R$, Lagrange multipliers $\nu_1 : \mathrm{T} \to R$, and/or $\nu_2 : \mathrm{T} \to R$, variables $\zeta(t_i$ at the times of discontinuity t_i for $\lambda(\cdot)$, and three constants α and β such that:

1. for every $t \in [0,T)$

$$\begin{aligned} L^*_u(t) &= 0\\ \dot{\lambda} &= -L^*_y(t)\\ \dot{y} &= L^*_\lambda\ (t) = G(t,y^*(t),u^*(t))\\ \nu_1(t) \geq 0, \text{ and } \nu_1(t)\,Q^*_1(t) &= 0\\ \nu_2(t) \geq 0, \text{ and } \nu_2(t)\,Q^*_2(t) &= 0 \end{aligned}$$

2. at the terminal time $t = T$

$$\begin{aligned} \lambda(T^-) &= \alpha\,Q^*_{2,y}\ (T) + \beta\,R^*_y\ (T)\\ \alpha \geq 0, &\text{ and } \alpha\,Q^*_2(T) = 0\\ \beta \geq 0, &\text{ and } \beta\,R^*(T) = 0. \end{aligned}$$

3. for any time t_i at a boundary interval and for any contact time t_i, the co-state variable $\lambda(\cdot)$ may have a discontinuity given by

$$\begin{aligned} \lambda(t_i^-) &= \lambda(t_i^+) + \eta(t_i)\,Q^*_{2,y}(t_i)\\ H^*(t_i^-) &= H^*\ (t_i^+) + \eta(t_i)\,Q^*_{2,t}(t_i)\\ \eta(t_i) \geq 0, &\text{ and } \eta(t_i)\,Q^*_2(t_i) = 0 \end{aligned}$$

   where $\lambda(t_i^-) = \lim_{t\uparrow t_i}\ \lambda(t)$ and $\lambda(t_i^+) = \lim_{t\downarrow t_i}\ \lambda(t)$ and an analogous notation for $H^*(t)$.

References: Köhler (1980), Hartl et al. (1995).

9.5 Integral constraints

In this section we present an optimal control problem in which there is a integral constraint of the iso-perimetric type, that is a constraint involving the integration of a known function of the state, and or control variables for all the domain of the independent variable. This case should not be confused to other types of optimal control problems in which there are integral constraints only for a sub-domain of the independent variable. While in the case we deal here, the constraint is of dimension zero (it is a scalar) in the second case the constraint is infinite-dimensional. This means that while in the case we address here we associate an adjoint variable, in the second case we to introduce an adjoint function.

Let the the independent variable be $x \in \mathrm{X} \subseteq \mathbb{R}$, where $\mathrm{X} = [x_0, x_1]$, the state variable be $y : \mathrm{X} \to\to \mathbb{R}$, and the control variable be $u : \mathrm{X} \to \mathbb{R}$.

We consider the following constraints

$$\int_{x_0}^{x_1} G_0\left(x, y(x), u(x)\right) dx \leq \bar{G} \tag{9.10a}$$

$$\frac{dy(x)}{dx} = G_1\left(x, y(x), u(x)\right) \; x \in \mathrm{X} \tag{9.10b}$$

$$x_0, x_1, y(x_0), y(x_1) \text{ free} \tag{9.10c}$$

where $\bar{G}$ is a constant.

The problem is

$$\begin{aligned} &\max_{x_0, x_1, u(\cdot)} \int_{x_0}^{x_1} F(x, y(x), y^{'}(x))dx \\ &\text{subject to} \\ &\int_{x_0}^{x_1} G_0(x, y(x), u(x))dx \leq \bar{G} \\ &\frac{dy(x)}{dx} = G_1(x, y(x), u(x)), \;\; x \in \mathrm{X} \\ &x_0, x_1, y(x_0), y(x_1) \text{ free} \end{aligned} \tag{ICP}$$

This problem optimal control problem has one functional constraint of the iso-perimetric type, (9.10a), one ordinary differential equation constraint, (9.10b), and has free initial and terminal indices and free initial and terminal values for the state variable .

There are several versions for this problem. For instance: (1) the simplest problem is the one in which x_0, x_1, $y(x_0)$ and $y(x_1)$ are fixed; (2) the free terminal problem which is common in optimal control problems in which the index variable is time in which x_0 and $y(x_0)$ are known and x_1 and $y(x_1)$ are free; (3) a problem in which the limit values of the indices, x_0 and x_1, are fixed and the state values, $y(x_0)$ and $y(x_1)$, are free; or (4) a problem in which the limit values of the indices, x_0 and x_1, are free and the state values, $y(x_0)$ and $y(x_1)$, are fixed.

We define the Hamiltonian function

$$H\left(x, y, u, \lambda_0, \lambda_1\right) = F\left(x, y, u\right) - \lambda_0 G_0\left(x, y, u\right) + \lambda_1 G_1\left(x, y, u\right)$$

where λ_0 is a constant, and λ_1 is a mapping $\lambda_1 : \mathrm{X} \to \mathbb{R}$. The maximized Hamiltonian, is

$$H^*(x) = H(x, y^*(x), y^*(x), \lambda_0, \lambda_1(x)) = \max_{u(\cdot)} \; H\left(x, y(x), u(x), \lambda_0, \lambda_1(x)\right), \text{ for each } \; x \in \mathrm{X}.$$

Proposition 2. ***[First order necessary conditions for the integral-constrained optimal control problem]*** *Let* (y^*, u^*) *be a solution to the OC problem* (ICP). *Then there is a variable* λ_0 *and a piecewise continuous function* $\lambda : \mathrm{X} \to \mathbb{R}$ *, such that* $(y^*, u^*, \lambda_0, \lambda_1)$ *satisfy the following conditions:*

- *the optimality condition:*
$$H^*_u(x) = 0, \; for\, x \in [x^*_0, x^*_1] \tag{9.11}$$
- *the multiplier equation*
$$\lambda^{'}_1(x) + H^*_y(x) = 0, \; for\, x \in [x^*_0, x^*_1] \tag{9.12}$$
- *initial and terminal conditions associated to the independent and state variables*
$$\lambda_1(x^*_j) = 0, \; for\, j = 0, 1 \tag{9.13a}$$
$$H^*(x^*_j) - \lambda_j(x^*_j)\, y^{*\prime}(x^*_j) = 0, \; for\, j = 0, 1 \tag{9.13b}$$
- *for admissible solutions, i.e., satisfying*
$$\int_{x^*_0}^{x^*_1} G_0\left(x, y^*(x), u^*(x)\right) dx = \bar{G} \tag{9.14a}$$
$$y^{*\prime}(x) = G_1\left(x, y^*(x), u^*(x)\right) x \in (x^*_0, x^*_1) \tag{9.14b}$$

Proof. The value functional at the optimum is

$$\mathsf{J}^* \; = \int_{x^*_0}^{x^*_1} F\left(x, y^*(x), u^*(x)\right) dx. \tag{9.15}$$

Equivalently, substituting the definition of the Hamiltonian and using integration by parts

$$\begin{aligned} \mathsf{J}^* &= \int_{x^*_0}^{x^*_1} \left(H\left(x, y^*(x), u^*(x), \lambda_0, \lambda_1(x)\right) - y^{*\prime}(x)\right) dx + \lambda_0 \bar{G} \\ &= \int_{x^*_0}^{x^*_1} \Big(H(x, y^*(x), u^*(x), \lambda_0, \lambda_1(x)) + \lambda^{'}_1(x) y^*(x)\Big)\, dx + \lambda_1(x^*_1) y^*(x^*_1) - \lambda_1(x^*_0) y^*(x^*_0) + \lambda_0 \bar{G} \end{aligned}$$

Introduce the arbitrary (functional) perturbations $y^*(x) \to y(x) = y^*(x) + \varepsilon \eta_y(x)$, $u^*(x) \to u(x) = u^*(x) + \varepsilon \eta_u(x)$, and the (point) perturbations $x^*_t \to x_t = x^*_t + \varepsilon \chi_t$, for $t = 0, 1$ and $y^*_t \to y_t = y^*_t + \varepsilon \iota_t$, for $t = 0, 1$, such that

$$\iota(x^*_t) = \iota_t - y^{'}(x^*_t)\, \chi_t, \; t = 0, 1 \tag{9.16}$$

At the optimum $\delta \mathsf{J}[y^*, u^*] = 0$ where the variational derivative is

$$\delta \mathsf{J}[y^*, u^*] = \lim_{\epsilon \to 0} \frac{\Delta \mathsf{J}}{\epsilon}$$

where $\Delta\mathsf{J} = \mathsf{J}[y^* + \varepsilon\iota, u^* + \varepsilon\eta_u] - \mathsf{J}[y^*, u^*]$. Using derivations from the previous problems we find

$$\begin{aligned}\Delta\mathsf{J}[y,u] = \int_{x_0^*}^{x_1^*} & \Big[H\left(x, y^*(x) + \varepsilon\eta_y(x), u^*(x) + \varepsilon\eta_u(x), \lambda_0, \lambda_1(x)\right) - H\left(x, y^*(x), u^*(x), \lambda_0, \lambda_1(x)\right) + \\ & +\lambda_1^{'}(x)\left(y^*(x) + \varepsilon\eta_y(x) - y^*(x)\right)\Big]\ dx+ \\ & + \lambda_1(x_1^*)\left(y^*(x_1^*) + \varepsilon\eta_y(x_1^*)\right) - \lambda_1(x_0^*)\left(y^*(x_0^*) + \varepsilon\eta_y(x_0^*)\right) - \lambda_1(x_1^*)y^*(x_1^*) + \lambda_1(x_0^*)y^*(x_0^*) + \\ & + \Big(H\left(x, y^*(x), u^*(x), \lambda_0, \lambda_1(x)\right)\big|_{x=x_1^*}\Big)\chi_1 - \Big(H\left(x, y^*(x), u^*(x), \lambda_0, \lambda_1(x)\right)\big|_{x=x_0^*}\Big)\chi_0\end{aligned}$$

Using a first-order Taylor approximation and equation (9.16), collecting terms, factoring out and simplifying the notation we have,

$$\begin{aligned}\delta\mathsf{J}[y,u] &= \int_{x_0^*}^{x_1^*} \Big[\ H_u^*(x)\eta_u(x) + \left(H_y^*(x) + \lambda_1^{'}(x)\right)\eta_y(x)\Big]\ dx + \lambda_1(x_1^*)\eta_y(x_1^*) - \lambda_1(x_0^*)\eta_y(x_0^*) + \\ & + H^*(x_1^*)\chi_1 - H^*(x_0^*)\chi_0\ = \\ &= \int_{x_0^*}^{x_1^*} \Big[\ H_u^*(x)\eta_u(x) + \left(H_y^*(x) + \lambda_1^{'}(x)\right)\eta_y(x)\Big]\ dx + \\ & + \lambda_1(x_1^*)\iota_1 - \lambda_1(x_0^*)\iota_0 + \left(H^*(x_1^*) - \lambda_1(x_1^*)(y^*)^{'}(x_1^*)\right)\chi_1 - \left(H^*(x_0^*) - \lambda_1(x_0^*)(y^*)^{'}(x_0^*)\right)\chi_0,\end{aligned}$$

at the optimum $\delta\mathsf{J}\ [y^*, u^*] = 0$ from which we derive equations (9.11)-(9.13b).

□

9.6 References

The definitive textbook on optimal control is Grass et al. (2008)

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