AME 2021-2022:
Problem set 4: Calculus of variations

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1 Calculus of variations

1. Solve the intertemporal optimization problem for a consumer, in an Arrow-Debreu exchange economy, where the utility function is $\int_0^\infty \ln(C(t))e^{-\rho t}dt$ where Arrow-Debreu prices and the endowment, at time t, are given by $P(t) = e^{-rt}$ and $Y(t) = Y(0)e^{\gamma t}$. The intertemporal budget constraint is $\int_0^\infty P(t)(Y(t) - C(t))dt = 0$.

   b) Find the first-order conditions for optimality (hint: this is a iso-perimetric problem).

   a) Solve the problem.

2. Consider the the intertemporal optimization problem for a consumer, in a finance economy, where the utility function is $\int_0^\infty \ln(C(t))e^{-\rho t}dt$. Assume that the instantaneous budget constraint, at time $t \geq 0$ is $\dot{A} = rA + Y - C$, where A is the stock of financial wealth, and r is the rate of return and Y is non-financial income. All the parameters (ρ, r, Y) are positive and constant. Let $A(0) = A_0$ and $\lim_{t\to\infty} e^{-rt}A(t) = 0$.

   a) Transform the problem into a calculus of variations problem and write the first order conditions

   b) Find the solution.

3. Assume a consumer problem in a finance economy, as in the previous example, in which the instantaneous utility function is CARA (constant absolute risk aversion) utility function. That is, consider the model:

$$\max_C \int_0^\infty -\frac{e^{-\theta C(t)}}{\theta} e^{-\rho t} dt,$$

   where ρ and θ are strictly positive. The restriction is $\dot{A} = rA - C$, given $A(0) = A_0 > \frac{r-\rho}{\theta r^2}$ and $\lim_{t\to\infty} e^{-rt}A(t) \geq 0$.

   a) Transform the problem into a calculus of variations problem and write the first order conditions

b) Find the solution.

4. Assume a AK model with a CARA (constant absolute risk aversion) utility function. That is, consider the model:

$$\max_{C} \int_0^{\infty} -\frac{e^{-\theta C(t)}}{\theta} e^{-\rho t} dt,$$

where ρ and θ are strictly positive, subject to the restriction $\dot{K} = AK(t) - C(t)$, given $K(0) = K_0 > \frac{A-\rho}{\theta A^2}$ and $\lim_{t\to\infty} e^{-At} K(t) \geq 0$.

   a) Transform the problem into a calculus of variations problem and write the first order conditions

   b) Find the solution.

5. Let the market value of the firm be given by the present value of cash-flows, $\int_0^{\infty} \left(Ak(t) - i(t)^2 \right) e^{-rt} dt$, where k is the capital stock, i is gross investment, r is the constant rate of interest and A is a productivity parameter. The capital accumulation is characterized by the equation $\dot{k} = i - \delta k$, where $\delta > 0$ is the capital depreciation rate. The initial capital stock is, $k(0) = k_0$, is known and we require that capital is bounded asymptotically by the solvability condition $\lim_{t\to\infty} e^{-rt} k(t) \geq 0$.

   (a) Transform the problem into a calculus of variations problem. Write the optimality conditions.

   (b) Find the solution to the problem. Provide an intuition for your results.