AME 2021-2022:
Problem set 7: Stochastic differential equations

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1 General

1. The diffusion equation is

$$dX(t) = \gamma X(t)dt + \sigma X(t)dW(t)$$

for $X(0) = x_0$.

(a) Prove that the solution is $X(t) = x_0 e^{(\gamma-\sigma^2/2)t+\sigma W(t)}$

(b) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

(c) Derive the backward Kolmogorov equation for the probability for $X(T) \leq 2x$ assuming that $X(t) = x$

(d) Derive the forward Kolmogorov equation for the density associated to $X(t) = x > 0$, assuming that $X(0) = 0$.

2. Consider the diffusion equation

$$dX(t) = \gamma X(t)dt + \sigma dW(t)$$

where $\{W(t)\}$ is a standard Brownian motion.

(a) Let $X(0) = x_0$ be known. Find the solution of the initial value problem.

(b) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

(c) Write the forward Kolmogorov equation for the density associated to $X(t) = x$. Provide an intuition for this equation.

3. Consider the diffusion equation

$$dX(t) = (1 - X(t))dt + dW(t),\ t \in [0, \infty)$$

where $\{W(t)\}$ is a standard Wiener process.

(a) Let $X(0) = 0$. Find the solution of the initial value problem.

(b) Find $\mathbb{E}[X(t)|X(0) = 0]$ and $\mathbb{V}[X(t)|X(0) = 0]$.

4. Consider the diffusion equation

$$dX(t) = -X(t)dt + dW(t),\ t \in [0, \infty)$$

where $\{W(t)\}$ is a standard Wiener process.

(a) Let $X(0) = x_0$, where x_0 is a real number. Find the solution to the initial value problem.

(b) Write the forward Fokker-Planck-Kolmogorov (FPK) for the density associated to $X(t) = x$, conditional on $X(0) = x_0$, that is $p(t, x) = \mathbb{P}[X(t) = x\,|X(0) = x_0]$.

(c) Let $P(t, s) = \mathcal{F}[p(t, x)]$ be the Fourier transform of $p(t, w)$. Find $P(t, s)$, which is the solution to the transformed FPK equation together with the initial condition $P(0, s) = \mathcal{F}[\delta(x - x_0)] = e^{-2\,\pi\,i\,s\,x_0}$(tip: $\mathcal{F}[x\,\partial_x p(t, x)] = -\big(P(t, s) + s\,\partial_s P(t, s)\big)$ and $\mathcal{F}[\partial_{xx} p(t, x)] = -(2\,\pi\,s)^2\,P(t, s)$).

5. The Vasicek 1977 (or Ornstein-Uhlenbeck) process is the solution of the equation

$$dX(t) = \theta(\mu - X(t))dt + \sigma dW(t)$$

for $X(0) = x_0$.

(a) Prove that the solution is

$$X(t) = \mu + (x_0 - \mu)e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s)} dW(s)$$

(b) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

2 Economic applications

1. Assume an AK model where $Y = A(t)K(t)$ where the productivity follows a SDE

$$dA(t) = \gamma dt + \sigma dW(t)$$

and the equilibrium equation is $dK(t) = sY(t)dt$, and $K(0) = k_0$ given. All the parameters, γ, σ and s are positive.

(a) Find the solution to the capital process $(K(t))_{t\in\mathbb{R}_+}$.

(b) Write the forward Fokker-Planck-Kolmogorov (FPK) for the density associated to $K(t) = k$, conditional on $K(0) = k_0 > 0$, that is $p(t, k) = \mathbb{P}[K(t) = k\,|K(0) = k_0]$.

(c) By solving the FPK equation find the conditional mean and variance of $K(t)$.

2. Consider a stochastic Solow model in which the population is constant and the production function is $Y(t) = A(t)K(t)^{\alpha}L^{1-\alpha}$ with $=0 < \alpha < 1$. The total factor productivity follows the process

$$dA(t) = \gamma A(t)dt + \sigma A(t)dW(t).$$

where $\gamma > 0$ and $\sigma > 0$ and $(W(t))$ is a Wiener process. The equilibrium in the product market is given by equation

$$\frac{dK(t)}{dt} = sY(t) - \delta K(t)$$

where $0 < s < 1$ is the savings rate and δ is the depreciation rate.

(a) Write the Itô stochastic differential equation for capital in intensive terms $k(t) \equiv K(t)/L(t)$

(b) Assuming that $K(0) = k_0$ and defining the conditional probability as $p(t, k) = \mathbb{P}[K(t) = k|K(0) = k_0]$ write the Fokker-Planck-Kolmogorov equation.

(c) Assume that the SDE you have found in (a) has the form $dk = \mu(k)dt + \sigma(k)dW(t)$. Consider an approximation of this SDE by the SDE $dk = \lambda\,(\bar{k} - k)\,dt + \sigma(k)dW(t)$, where $\bar{k}$ is the steady state of the squeleton (i.e., the value of $k > 0$ such that $\mu(k) = 0$ and $\lambda = \frac{\partial \mu}{\partial k}(\bar{k})$ (the derivative evaluated at that steady state). Find the Fokker-Planck-Kolmogorov equation.

(d) Find the asymptotic moments of the process of the approximated process and provide an intuition for your results.

3 Optimal control of stochastic differential equations

1. For a given initial level of the capital stock $K(0) = k_0$, the stock of capital, $K(t)$, of a firm evolves according to the diffusion process

$$dK(t) = \big(I(t) - \delta\, K(t)\big)dt + \sigma\, dW(t),\ t > 0$$

where $I(t)$ is gross investment, $\delta > 0$ is the capital depreciation rate, and $\sigma\, dW(t)$ represents additive random shocks to the capital accumulation process, where $\{W(t)\}_{t\in\mathbb{R}_+}$ is a Wiener process.

The firm's objective is to use gross investment to maximize the expected value of the present value of the cash flows,

$$\mathbb{E}_0\left[\int_0^{\infty}\Big(AK(t) - \frac{I(t)^2}{2}\Big)e^{-rt}dt\right].$$

(a) Write the Hamilton-Jacobi-Bellman equation.

(b) Solve the HJB equation (hint: use a linear trial value function).

(c) Obtain the stochastic differential equation (SDE) for the optimal capital accumulation equation.

(d) Solve that equation. **Remark**: if you could not obtain the SDE in (c), use instead $dK(t) = \big(\bar{I} - \delta\, K(t)\big)dt + \sigma\, dW(t)$ where $\bar{I} > 0$ is a constant, in this question and in the following questions.

(e) Find the statistics for the optimal capital stock of the firm (expected value and variance).

(f) Provide an intuitive discussion of your results (hint: compare them with the deterministic analog).

(g) Write the Fokker-Planck-Kolmogorov equation associated to the distribution of the capital stock, starting from $K(0) = k_0$, which is fully observed.

(h) Solve this equation for the case in which $\delta = 0$.

2. Consider the following stochastic resource-depletion problem, where $\{X(t)\}_{t\in\mathbb{R}}$ is the process for the stock of the resource, and $\{C(t)\}_{t\in\mathbb{R}}$ is the process for its use,

$$\max_{C(\cdot)} \mathbb{E}_0\Big[\int_0^\infty \ln\left(C(t)\right) e^{-\rho\, t}\Big]$$

subject to

$$dX(t) = -C(t)\, dt + \sigma\, X(t)\, dW(t), \text{ for } t \in (0, \infty)$$

where $\{W(t)\}_{t\in\mathbb{R}}$ is a Wiener process, and $X(0) = x_0 > 0$ is given. The rate of time preference and the volatility parameters, ρ and σ, are both positive and satisfy $\rho > \sigma^2$.

(a) Write the first-order conditions for optimality according to the stochastic Pontriyagin's maximum principle.

(b) Find the stochastic process for $C(t)$.

(c) Using $C(t) = \phi\, X(t)$, for an undetermined constant ϕ , as a trial function find the solution for the optimal $X(t)$.

3. Consider the problem

$$\max_{C,\theta} \mathbb{E}_0\Big[\int_0^\infty \ln\left(C(t)\right) e^{-\rho\, t} dt\Big],\ \rho > 0$$

subject to

$$dN = \Big(\big(r\,(1-\theta) + r^s\,\theta\big)\, N + y - C\Big)dt + \sigma\,\theta\, dW(t)$$

where we consider the process for consumption, C, net financial wealth, N, the portfolio composition, θ, and the Wiener process. The initial wealth $N(0) = n_0$ is given and wealth is asymptotically bounded. Solve the stochastic problem for a representative consumer assuming a log utility function, for the case in which the non-financial income is constant.

(a) Using the principle of dynamic programming.

(b) Using the stochastic Pontriyagin maximum principle.

References

Vasicek, O. A. (1977). “An equilibrium characterization of the term structure”. In: *Journal of Financial Economics* 5, pp. 177–88.