Universidade de Lisboa
Instituto Superior de Economia e Gestão

PhD in Economics
Advanced Mathematical Economics
2021-2022

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Exam: **Exam** (Época Normal)
21.1.2022 (18.00h-21.00h)

Instructions:

- This is an open book exam. The use of any electronic device is forbidden.
- Your answer should be concise, objective and specific. Any notation, calculation, motivation, discussion or explanation **not strictly related to the specific question** it tries to address will either not be considered or have a negative assessment.

1. `[2 points (1,1)]` The government budget constraint is $\dot{B} = D(t) + iB$, where B is the net stock of government bonds. The (time-varying) primary deficit and the (constant) interest rate on government are represented by $D(t)$ and i, respectively. Assume that $i > 0$ and $D(\cdot)$ can take any value.
   (a) Assuming that the sustainability condition is $\lim_{t\to\infty} e^{-i\,t}B(t) = 0$, find the sustainable initial value of the net debt, $B(0)$.
   (b) Let $D(t) = D$ be constant. Introduce monetary financing, and a financing rule such that $M(t) = \theta\,(M(t) + B(t))$, for every $t \in [0, \infty)$, where $0 < \theta < 1$ is constant. Is the initial level of the government's net debt that you determined in (a) still sustainable ? (hint: note that the government budget constraint is now $\dot{B} + \dot{M} = D + iB$).

2. `[5 points (2,1.5,1.5)]` Consider the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$ where $\mathbf{y} \in \mathbb{R}^2$

$$\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

   (a) Solve the ODE.
   (b) Draw the phase diagram. Justify it analytically.
   (c) Assume the initial value $y_1(0) = 2$. Find the solution(s) converging to the steady state.

3. `[3 points (1.5,1.5)]` Consider the ODE $\dot{y} = (a - y^2)\,(b + y^2)$, for $y : \mathbb{R}_+ \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$.
   (a) Find the steady states for different values of a and b (suggestion: draw a two-entry table in (a, b) containing the different possible cases).
   (b) Assume that $b = 0$. Draw the bifurcation diagram, and the representative phase diagrams in $(y, \dot{y})$-diagrams.

4. `[6 points (2, 2, 1,1)]` Consider the following benchmark advertising problem for a firm (as in Nerlove and Arrow (1962)). The profit of the firm is increasing in the costumer awareness (also called goodwill or customer base) of the firm's product, denoted by A. The costumer base is subject to a natural decay, which can be explained by forgetfulness by customers. The firm both increases its costumer base and counters the customer forgetfulness by advertising its product. However, advertising increases the costumer awareness with some friction (its effect is slow) and also entails costs. Formally, denoting by a the intensity of advertising, the problem is

$$\max_{a(\cdot)} \int_0^\infty e^{-r\,t}\Big(\pi(A(t)) - c(a(t))\Big)\,dt$$
$$\text{subject to}$$
$$\dot{A} = a - \delta\,A$$
$$A(0) = a_0 \text{ given}$$

were r is the interest rate and δ is the costumers' rate of forgetfulness. The sales dependence on awareness is assumed to be linear, $\pi(A) = \mu\,A$, and the advertising cost function is $c(a) = (1+\xi)\,a^{1+\xi}$. All the parameters, r, δ, μ and ξ are positive.

(a) Write the optimality conditions, and obtain the MHDS in (A, a).

(b) Find the steady state(s) and characterize its (their) dynamic properties.

(c) Draw the phase diagram.

(d) What are the effects of an unanticipated and permanent increase in the rate of interest, r ? Provide a brief intuition.

5. `[4 points (2,1,1)]` Consider the stochastic differential equation

$$dX(t) = (a - X(t))dt + b\,dW(t),\ t \in [0, \infty)$$

where $\Big(W(t)\Big)_{t\geq 0}$ is a standard Wiener process, and a and b are positive constants.

(a) Let $X(0) = a$. Find the solution of the initial value problem (hint: use the transformation $Y(t) = X(t)\,e^t$).

(b) Write the Fokker-Planck-Kolmogorov equation for $\mathbb{P}[X(t) = x|X(0) = a]$. Solve it.

(c) Find $\mathbb{E}[X(t)|X(0) = a]$ and $\mathbb{V}[X(t)|X(0) = a]$. Provide an intuition for their asymptotic values (i.e., for their limit when $t \to \infty$).

References

Nerlove, M. and Arrow, K. J. (1962). Optimal advertising policy under dynamic conditions. *Economica*, 29(114):pp. 129–142.