Universidade de Lisboa
Instituto Superior de Economia e Gestão

PhD in Economics
**Advanced Mathematical Economics**
2021-2022

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Exam: **Re-sit exam** (Época Recurso)
3.2.2022 (18.00h-21.00h)

**Instructions**:

- This is an open book exam. The use of any electronic device is forbidden.
- Your answer should be concise, objective and specific. Any notation, calculation, motivation, discussion or explanation **not strictly related to the specific question** it tries to address will either not be considered or have a negative assessment.

1. `[2 points (1,1)]` A utility function, $u(x)$, is said to display hyperbolic absolute risk aversion if it satisfies
$$\frac{u''(x)}{u'(x)} = \frac{a}{b+x},\ x \in \mathbb{R}_+$$
where $a$ and $b$ are real numbers.
   (a) Find the general solution to the ODE.
   (b) Prove that if $a \equiv \gamma - 1$, $b \equiv \frac{\eta\,(1-\gamma)}{\beta}$, and $u'(0) = \frac{\beta\,\gamma}{\eta\,(1-\gamma)}\,u(0)$ we can find the benchmark case presented in Merton, R. (1971). Optimum consumption and portfolio rules in a continuous time model. *Journal of Economic Theory*, 3:373–413
$$u(x) = \frac{1-\gamma}{\gamma}\left(\frac{\beta}{1-\gamma}\,x + \eta\right)^{\gamma}.$$
   Find $u(0)$.

2. `[5 points (1.5,1.5,1.5,0.5)]` Consider the planar ODE for $\mathbf{y} : \mathbb{R}_+ \to \mathbb{R}^2$, with the independent variable $t \in \mathbb{R}_+$,
$$\begin{aligned}\dot{y}_1 &= -y_1 + 2\,y_2 + \alpha,\\ \dot{y}_2 &= y_1 - 2\,y_2.\end{aligned} \tag{1}$$
   (a) Assume that $\alpha = 0$. Draw and characterize the phase diagram.
   (b) Assume that $\alpha = 0$. Solve the equation analytically.
   (c) Assume that $\alpha = 1$. Draw the phase diagram and characterize it.
   (d) Prove that the solution to the ODE of the equation (1) verifies $y_1(t)+y_2(t)-\alpha\,t = y_1(0)+y_2(0)$ for any $t \in \mathbb{R}_+$.

3. `[3 points (1,1,1)]` Consider the following non-linear planar ODE:

$$\dot{y}_1 = -y_1\,(1 + \lambda\, y_2),$$
$$\dot{y}_2 = y_2\,(1 - \lambda\, y_1),$$

for $(y_1, y_2) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

(a) For $\lambda > 0$, find the steady state(s) and characterize its (their) local dynamics.
(b) Draw the phase diagram for $\lambda > 0$.
(c) Provide a bifurcation diagram for all possible values of the parameter $\lambda$.

4. `[5 points (2,2,1)]` A household's problem, in which the participation in the asset market envolves adjustment costs, is

$$\min_{c(\cdot)} \int_0^\infty \frac{c(t)^{1-\theta} - 1}{1-\theta}\, e^{-\rho t} dt$$

subject to

$$\dot{a} = r\,a + w - c - \eta\,a^2$$
$$a(0) = a_0 \text{ given}$$
$$\lim_{t\to\infty} e^{-r\,t}\,a(t) \geq 0$$

where $\theta > 0$, $\eta > 0$ and $w > 0$.

(a) Write the first-order necessary conditions for optimality according to the Pontriyagin's maximum principle as a dynamic system in $(a, c)$. Are those conditions also sufficient ?
(b) Draw the phase diagrams for the three cases: (a) with $r < \rho$, (b) with $r = \rho$, and (c) with $r > \rho$.
(c) Let $r = \rho$. Obtain, geometrically and analytically the multipliers for a permanent and non-anticipated increase in $w$. Provide an intuition for your results.

5. `[2 points (0.5,1,0.5) ]` Consider the partial differential equation

$$\partial_t u(t, x) = a\,\partial_{xx} u(t, x) + d, \text{ for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}$$

where $a > 0$ and $d$ is an arbitrary number.

(a) Classify the PDE.
(b) Find the general solution to the PDE.
(c) If $u(0, x) = \delta(x - x_0)$, where $\delta(.)$ is Dirac's delta "function", and $x_0$ is a known number, find the solution to the initial-value problem. Provide an intuitive characterization of the solution.

6. `[3 points (1,1,1)]` Consider the following problem

$$dX(t) = \left(\frac{\sigma}{2}\right)^2 dt + \sigma\,\sqrt{X(t)}\,dW(t),\ t \in (0,\,T]$$
$$X(T) = a^2,$$

where $\{W(t)\}$ is a standard Brownian motion, and $\sigma$, $a$ and $T$ are positive numbers. In order to solve the problem, consider the following steps:

(a) Find the process for $Y(t) = \sqrt{X(t)}$ by using Itô's lemma.
(b) Find the general solution for $X(t)$.
(c) Find the particular solution to the proposed problem.