Advanced Mathematical Economics

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Contents

Chapter 3

Planar linear ODE

3.1 Introduction

In this chapter we deal the planar ordinary differential equation (ODE) over function $\mathbf{y} : \mathrm{X} \subseteq \mathbb{R} \rightarrow \mathrm{Y} \subseteq \mathbb{R}^2$ of type

$$\mathbf{F}(\nabla \mathbf{y}(x), \mathbf{y}(x), x) = \mathbf{0}.$$

The equation is planar because the range of $\mathbf{y}$ is two-dimensional, $\mathbf{y} \in \mathrm{Y} \subseteq \mathbb{R}^2$,

$$\mathbf{y}(x) \equiv \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

it is ordinary because the domain of the independent variable has dimension one, $x \in \mathbb{X} \subset \mathbb{R}$, and it is differential because it assumes a variational approach to modelling, that is it is a functional equation containing the gradient

$$\nabla \mathbf{y} \equiv \begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} = \begin{pmatrix} \frac{dy_1(x)}{dx} \\ \frac{dy_2(x)}{dx} \end{pmatrix}.$$

In this chapter we will consider the following case:

Definition 1. *A* ***planar linear autonomous ordinary differential equation*** *is a functional equation is the following equation: in matrix form*

$$\nabla \mathbf{y}(x) = \mathbf{A}\, \mathbf{y}(x) + \mathbf{B} \tag{3.1}$$

where the coefficient matrices $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and $\mathbf{B} \in \mathbb{R}^2$ have constant elements,

$$\mathbf{A} \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ \mathbf{B} \equiv \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{3.2}$$

or in expanded form

$$\begin{aligned} y_1'(x) &= a_{11}\, y_1(x) + a_{12}\, y_2(x) + b_1 \\ y_2'(x) &= a_{21}\, y_1(x) + a_{22}\, y_2(x) + b_2. \end{aligned} \tag{3.3}$$

Furthermore, if $\mathbf{B} = \mathbf{0}$ *then the ODE is called* ***homogeneous****, and if* $\mathbf{B} \neq \mathbf{0}$ *it is called* ***non-homogeneous****.*

As with the scalar linear ODE, equation (3.1) (or in form (3.3)) has explicit solutions. However, given its dimension the solutions are more complex. In this chapter we present the general solutions of ODE (3.1) for any independent variable. In the next chapter we consider the case in which the independent variable is time and present the important results on the dynamics that can be generated by a time-dependent ODE.

The content of the chapter is the following: in section 3.2 we review some useful algebra results, in section 3.3 we derive the matrix exponential function. In sections 3.4 and 3.5 we solve the homogeneous and non-homogeneous ODE, respectively.

3.2 Two dimensional matrix algebra results

Matrix $\mathbf{A}$, in equation (3.2) fundamentally determines the solution to differential equation (3.1). It also allows for the characterization of its dynamics as we will see in the next chapter.

It is possible to classify any matrix $\mathbf{A}$ as being:

1. a **canonical matrix** similar to one of the following three matrices, called the **Jordan canonical forms**[1]

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix},\ \mathbf{\Lambda}_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},\ \text{or } \mathbf{\Lambda}_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \tag{3.4}$$

   belonging to $\mathbb{R}^{2\times 2}$, because λ_-, λ_+ α and β are real numbers. Matrix $\mathbf{\Lambda}_3$ can also be written as

$$\mathbf{\Lambda}_3^c = \begin{pmatrix} \alpha - \beta\, i & 0 \\ 0 & \alpha + \beta\, i \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

   where $i = \sqrt{-1}$ is the imaginary number.

2. or, a **non-canonical matrix** if is of one of the two following forms

$$\mathbf{A}_d \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},\ \text{or } \mathbf{A}_h \equiv \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \tag{3.5}$$

   where λ, α and β are real numbers.

Two matrices are said to be **similar** if they have the same spectrum. The **spectrum of matrix** $\mathbf{A}$ is a tuple belonging to $\mathbb{C}^2$ (the space of two-dimensional complex numbers)

$$\sigma(\mathbf{A}) = \left\{ \lambda \in \mathbb{C}^2 : \det\left(\mathbf{A} - \lambda\,\mathbf{I}\right) = 0 \right\}.$$

[1] See the appendix 3.A.1 where we gather some useful results from matrix algebra.

where $\mathbf{I}$ is the (2×2) identity matrix.

The elements of $\sigma(\mathbf{A})$ are called the **eigenvalues** of $\mathbf{A}$.

In order to determine the spectrum, we need to find the **characteristic polynomial** associated to matrix $\mathbf{A}$, which is the square polynomial in λ

$$\det(\mathbf{A} - \lambda\,\mathbf{I}) = \lambda^2 - \text{trace}(\mathbf{A})\ \lambda + \det(\mathbf{A}),$$

whose coefficients are the trace and the determinant of $\mathbf{A}$,

$$\text{trace}(\mathbf{A}) \ = a_{11} + a_{22},\ \text{and}\ \det(\mathbf{A}) \ = a_{11}\,a_{22} - a_{12}\,a_{21}.$$

Equation $\det(\mathbf{A} - \lambda\,\mathbf{I}) = 0$ is called **characteristic equation**. The eigenvalues of matrix $\mathbf{A}$ are the solutions to the characteristic equation:

$$\begin{cases} \lambda_- = \dfrac{\text{trace}(\mathbf{A})}{2} - \sqrt{\Delta(\mathbf{A})}, \\ \lambda_+ = \dfrac{\text{trace}(\mathbf{A})}{2} + \sqrt{\Delta(\mathbf{A})} \end{cases} \tag{3.6}$$

where

$$\Delta(\mathbf{A}) \equiv \left(\frac{\text{trace}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A})$$

is called the discriminant of matrix $\mathbf{A}$.

Eigenvalues of A

Finding the eigenvalues allows us to classify any matrix according to three criteria:

1. the sign of the discriminant allows us to determine if the eigenvalues are real or complex numbers, and to find the Jordan canonical form of matrix $\mathbf{A}$ we can call $\mathbf{\Lambda}$;
2. the sign of the trace and the determinant allows us to sign the eigenvalues if they are real or the sign of their real part if they are complex;
3. their genericity, i.e., the robustness of the classification provided by the previous two criteria to small change in the elements of $\mathbf{A}$

First, the two eigenvalues are real if $\Delta(\mathbf{A}) \geq 0$ and they are complex conjugate if $\Delta(\mathbf{A}) < 0$. In particular, if $\Delta(\mathbf{A}) > 0$ the eigenvalues are real and distinct and satisfy $\lambda_- < \lambda_+$, if $\Delta(\mathbf{A}) = 0$ the eigenvalues are real and multiple and satisfy $\lambda = \lambda_- = \lambda_+ = \dfrac{\text{trace}(\mathbf{A})}{2}$, and if $\Delta(\mathbf{A}) < 0$ they are complex conjugate and satisfy

$$\lambda_\pm = \alpha \pm \beta\, i,\ \text{for}\ i \equiv \sqrt{-1}$$

where $\alpha = \dfrac{\text{trace}(\mathbf{A})}{2}$ and $\beta = \sqrt{|\Delta(\mathbf{A})|}$.

Second, the signs of the real part of both eigenvalues is the same if $\det(\mathbf{A}) \ > 0$ and it is different if $\det(\mathbf{A}) \ < 0$. In the first case they are both positive if $\det(\mathbf{A}) \ > 0$ and $\text{trace}(\mathbf{A}) > 0$ and they are both negative if $\det(\mathbf{A}) \ > 0$ and $\text{trace}(\mathbf{A}) < 0$.

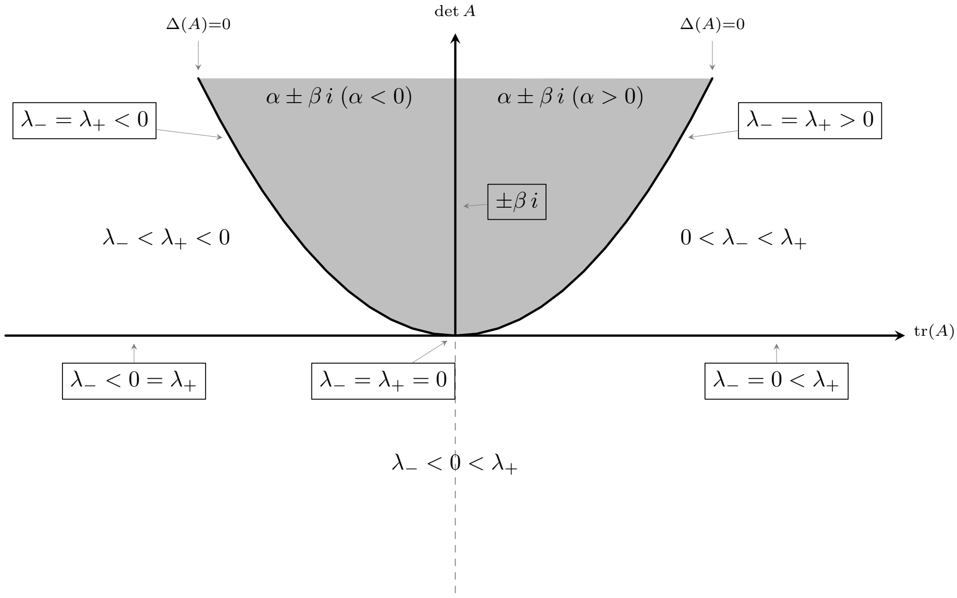


Figure 3.1: Eigenvalues of $\mathbf{A}$ in the $(\text{trace}(\mathbf{A}), \det(\mathbf{A}))$ space. The gray area corresponds to the existence of complex conjugate eigenvalues.

Third, the eigenvalues are generic in the sense that they will not change their type or sign for small changes in the elements of the coefficient matrix $\mathbf{A}$ if $\Delta(\mathbf{A}) \neq 0$, or $\det(\mathbf{A}) \neq 0$, or $\text{trace}(\mathbf{A}) \neq 0$ and $\det(\mathbf{A}) \geq 0$, and they are not generic otherwise, that is if $\Delta(\mathbf{A}) = 0$, or $\det(\mathbf{A}) = 0$, or $\text{trace}(\mathbf{A}) = 0$ and $\det(\mathbf{A}) \geq 0$.

Figure 3.1, shows all the possible relevant cases. There are five generic cases (corresponding to two-dimensional subsets), four non-generic cases of co-dimension-one (corresponding to lines) and two co-dimension-two case (the origin). It displays all the following cases:

1. the five generic cases are: (1) if $\text{trace}(\mathbf{A}) > 0$, $\det(\mathbf{A}) > 0$ and $\Delta(\mathbf{A}) > 0$ the two eigenvalues are real, different and positive, $\lambda_+ > \lambda_- > 0$; (2) if $\text{trace}(\mathbf{A}) > 0$, $\det(\mathbf{A}) > 0$ and $\Delta(\mathbf{A}) < 0$ the two eigenvalues are complex conjugate with positive real parts $\lambda_\pm = \alpha \pm \beta\, i$ with $\alpha > 0$; (3) if $\text{trace}(\mathbf{A}) < 0$, $\det(\mathbf{A}) > 0$ and $\Delta(\mathbf{A}) > 0$ the two eigenvalues are real, different, and negative $0 > \lambda_+ > \lambda_-$; (4) if $\text{trace}(\mathbf{A}) < 0$, $\det(\mathbf{A}) > 0$ and $\Delta(\mathbf{A}) < 0$ the two eigenvalues are complex conjugate with negative real parts, $\lambda_\pm = \alpha \pm \beta\, i$ with $\alpha < 0$; or (5) if $\det(\mathbf{A}) < 0$ the two eigenvalues are real and with opposite signs $\lambda_+ > 0 > \lambda_-$;

2. the six non-generic cases: (1) if $\text{trace}(\mathbf{A}) > 0$ and $\Delta(\mathbf{A}) = 0$ the two eigenvalues are real, equal and positive $\lambda_+ = \lambda_- > 0$; (2) if $\text{trace}(\mathbf{A}) < 0$ and $\Delta(\mathbf{A}) = 0$ the two eigenvalues are real, equal and negative $\lambda_+ = \lambda_- < 0$; (3) if $\text{trace}(\mathbf{A}) = 0$ and $\det(\mathbf{A}) > 0$ then the two eigenvalues are complex conjugate with zero real part, $\lambda_\pm = \pm\beta\, i$; (4) if $\text{trace}(\mathbf{A}) > 0$ and $\det(\mathbf{A}) = 0$ the two eigenvalues are real one is positive and the other is equal to zero, $\lambda_+ > 0 = \lambda_-$; (5) (4) if $\text{trace}(\mathbf{A}) < 0$ and $\det(\mathbf{A}) = 0$ the two eigenvalues are real one is

negative and the other is equal to zero, $\lambda_+ = 0 < \lambda_-$; or (6) if $\text{trace}(\mathbf{A}) = \det(\mathbf{A}) = 0$ both eigenvalues are real and equal to zero, $\lambda_+ = \lambda_- = 0$.

Therefore $\sigma(\mathbf{A}) \in \mathbb{R}^2$ if $\Delta(\mathbf{A}) \geq 0$ and $\sigma(\mathbf{A}) \in \mathbb{C}^2$ if $\Delta(\mathbf{A}) < 0$.

There is a useful result on the relationship between the coefficients of the characteristic equation with elementary operations between the eigenvalues of any matrix $\mathbf{A}$:

$$\lambda_- + \lambda_+ = \text{trace}(\mathbf{A}), \tag{3.7a}$$

$$\lambda_- \, \lambda_+ = \det(\mathbf{A}). \tag{3.7b}$$

Clearly, $\det(\mathbf{A}) < 0$ is a sufficient condition for the existence of two real eigenvalues and is a necessary and sufficient condition for $\lambda_- < 0 < \lambda_+$. This is a very useful result for economic models.

Canonical matrices

There is a close relationship between the discriminant of a matrix $\mathbf{A}$, which is not in a non-canonical form as in equation (3.5), and to its similar Jordan canonical form[2], which we call the **Jordan canonical form of A**.

Lemma 1. ***Jordan canonical form of a matrix* A** *The Jordan canonical form of* $\mathbf{A}$ *is determined by the sign of the discriminant* $\Delta(\mathbf{A})$*: if* $\Delta(\mathbf{A}) > 0$ *then the Jordan canonical form of* $\mathbf{A}$ *is* $\mathbf{\Lambda}_1$*, if* $\Delta(\mathbf{A}) = 0$ *the Jordan canonical of* $\mathbf{A}$ *is* $\mathbf{\Lambda}_2$*, and if* $\Delta(\mathbf{A}) < 0$ *the Jordan canonical form of* $\mathbf{A}$ *is* $\mathbf{\Lambda}_3$*.*

Given any matrix $\mathbf{A}$, and its Jordan canonical form, given in equation (3.4), the fundamental theorem of Algebra states that there is a (non-singular) linear operator $\mathbf{P} \in \mathbb{R}^{2\times 2}$ such that the following relationship holds

$$\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1} \Leftrightarrow \mathbf{\Lambda} = \mathbf{P}^{-1}\,\mathbf{A}\,\mathbf{P}. \tag{3.8}$$

Matrix $\mathbf{P}$ is called the **eigenvector matrix** associated to matrix $\mathbf{A}$.

The fact that any matrix $\mathbf{A}$ has a one-to-one relationship with one of the Jordan canonical forms allows us to reduce the determination of the general solution of a planar ODE to the solution of a simpler ODE in which the coefficient matrix is its Jordan canonical form. Next, we can transform back to the original ODE by using $\mathbf{P}$ as an operator.

Non-canonical matrices

For non-canonical matrices, represented in equation (3.5), the spectra are: first, in the case of matrix $\mathbf{A}_d$ there are multiple eigenvalues, $\sigma(\mathbf{A}_d) = \{\, \lambda, \lambda\}$, although the matrix is not of the form $\mathbf{\Lambda}_2$; and, second, in the case of matrix $\mathbf{A}_h$ the spectrum is $\sigma(\mathbf{A}_h) = \{\, \alpha + \beta, \alpha - \beta\}$ which are two real and distinct numbers.

[2]See the appendix 3.A.1.

3.3 The two-dimensional matrix exponential function

We saw that the (general) solution of the scalar linear homogeneous equation $y'(x) = a\,y$ is $y(x) = y(x_0)\,e^{ax}$ where $y(x_0)$ is an arbitrary element of $\mathrm{Y} \subseteq \mathbb{R}$ for $x = x_0 \in \mathrm{X}$. Recall that the exponential function has the series representation

$$e^{\lambda x} \equiv \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!} = 1 + \lambda x + \frac{1}{2}\,(\lambda x)^2 + \frac{1}{6}\,(\lambda x)^3 + ...$$

For the planar problem we can also define a **matrix exponential** function

$$\mathbf{e}^{\mathbf{A}\mathbf{x}} \equiv \sum_{n=0}^{+\infty} \frac{1}{n!}\mathbf{A}^n x^n = \mathbf{I} + \mathbf{A}x + \frac{1}{2}\,\mathbf{A}^2 x^2 + ... \tag{3.9}$$

which is a mapping $\mathbf{e}^{\mathbf{A}\mathbf{x}} : \mathrm{X} \to \mathbb{R}^{2\times 2}$ with the following properties:[3]

Lemma 2 (Properties of matrix exponentials $\mathbf{e}^{\mathbf{A}\mathbf{x}}$)**.** *Matrix exponential function* $\mathbf{e}^{\mathbf{A}\mathbf{x}}$*, defined in equation* (3.9) *has the following properties:*

(i) semigroup property: $\mathbf{e}^{\mathbf{A}(\mathbf{x}+\mathbf{s})} = \mathbf{e}^{\mathbf{A}\mathbf{x}}\mathbf{e}^{\mathbf{A}\mathbf{s}}$

(ii) inverse of the matrix exponencial is the the exponential of the inverse: $\left(\mathbf{e}^{\mathbf{A}\mathbf{x}}\right)^{-1} = \mathbf{e}^{-\mathbf{A}\mathbf{x}}$

(iii) the time derivative commutes: $\frac{d}{dx}\mathbf{e}^{\mathbf{A}\mathbf{x}} = \mathbf{A}\mathbf{e}^{\mathbf{A}\mathbf{x}} = \mathbf{e}^{\mathbf{A}\mathbf{x}}\mathbf{A}$

(iv) if matrices $\mathbf{A}$ *and* $\mathbf{B}$ *commute, (i.e., if* $\mathbf{A}\,\mathbf{B} = \mathbf{B}\,\mathbf{A}$*) then* $\mathbf{e}^{(\mathbf{A}+\mathbf{B})\mathbf{x}} = \mathbf{e}^{\mathbf{A}\mathbf{x}}\mathbf{e}^{\mathbf{B}\mathbf{x}}$

(v) Let $\mathbf{P}$ *be a non-singular and square matrix. Then*

$$\mathbf{e}^{\boldsymbol{\Lambda}\mathbf{x}} = \mathbf{e}^{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{x}} = \mathbf{P}^{-1}\mathbf{e}^{\mathbf{A}\mathbf{x}}\mathbf{P}.$$

From Lemma 2 *(v)* as $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \boldsymbol{\Lambda}$ then $\mathbf{e}^{\boldsymbol{\Lambda}\mathbf{x}} = \mathbf{e}^{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{x}} = \mathbf{P}^{-1}\mathbf{e}^{\mathbf{A}\mathbf{x}}\mathbf{P}$ or, equivalently

$$\mathbf{e}^{\mathbf{A}\mathbf{x}} = \mathbf{P}\,\mathbf{e}^{\boldsymbol{\Lambda}\mathbf{x}}\,\mathbf{P}^{-1}, \tag{3.10}$$

where $\boldsymbol{\Lambda}$ is the Jordan canonical form of $\mathbf{A}$.

Therefore, given any matrix $\mathbf{A}$, the exponential matrix $\mathbf{e}^{\mathbf{A}\mathbf{x}}$ is a (2×2) dimensional function of x. It depends on x because , and it depends on x because $\mathbf{e}^{\mathbf{A}\mathbf{x}}$ is a linear transformation of $\mathbf{e}^{\boldsymbol{\Lambda}\mathbf{x}}$ performed by the operator matrix $\mathbf{P}$.

This is an important result which means that the types of solutions, and the associated phase diagrams, can be completely enumerated.

The exponential matrices for the Jordan canonical forms are:

[3] See Hirsch and Smale (1974).

Lemma 3 (Matrix exponential functions for Jordan canonical forms)**.** *Let* $\mathbf{\Lambda}$ *be a matrix in an arbitrary Jordan canonical form, as in equation* (3.4)*, and let* λ_-*,* λ_+*,* λ*,* α *and* β *be real numbers. Then,*

- *If* $\mathbf{\Lambda} = \mathbf{\Lambda}_1$ *then*

$$\mathbf{e}^{\mathbf{\Lambda x}} = \mathbf{e}^{\mathbf{\Lambda_1 x}} = \begin{pmatrix} e^{\lambda_- x} & 0 \\ 0 & e^{\lambda_+ x} \end{pmatrix}. \tag{3.11}$$

- *If* $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ *then*

$$\mathbf{e}^{\mathbf{\Lambda x}} = \mathbf{e}^{\mathbf{\Lambda_2 x}} = e^{\lambda x} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \tag{3.12}$$

- *If* $\mathbf{\Lambda} = \mathbf{\Lambda}_3$ *then*

$$\mathbf{e}^{\mathbf{\Lambda x}} = \mathbf{e}^{\mathbf{\Lambda_3 x}} = e^{\alpha x} \begin{pmatrix} \cos \beta x & \sin \beta x \\ -\sin \beta x & \cos \beta x \end{pmatrix} \ \ or \ \ \begin{pmatrix} e^{(\alpha+\beta i)x} & 0 \\ 0 & e^{(\alpha-\beta i)x} \end{pmatrix} \tag{3.13}$$

Proof. Consider the definition of matrix exponential, equation (3.9) and the Jordan canonical form matrices in equation (3.4). In the first case, we have

$$\mathbf{e}^{\mathbf{\Lambda_1 x}} = \mathbf{I}_2 + \mathbf{\Lambda}_1 x + \frac{1}{2}\,(\mathbf{\Lambda}_1)^2 x^2 + ... = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda_- x & 0 \\ 0 & \lambda_+ x \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \lambda_-^2 x^2 & 0 \\ 0 & \lambda_+^2 x^2 \end{pmatrix} + ...$$

then, performing the matrix additions,

$$\mathbf{e}^{\mathbf{\Lambda_1 x}} = \begin{pmatrix} 1 + \lambda_- x + \frac{1}{2}\lambda_-^2 x^2 + ... & 0 \\ 0 & 1 + \lambda_+ x + \frac{1}{2}\lambda_+^2 x^2 + ... \end{pmatrix} = \begin{pmatrix} e^{\lambda_- x} & 0 \\ 0 & e^{\lambda_+ x} \end{pmatrix}$$

because $e^y = \sum_{n=0}^{+\infty} \frac{y^n}{n!}$. That result is straightforward to obtain because the Jordan matrix is diagonal. This is not the case for Jordan matrix $\mathbf{\Lambda}_2$, though. But if we decompose $\mathbf{\Lambda}_2$ as

$$\mathbf{\Lambda}_2 = \mathbf{\Lambda}_{2,1} + \mathbf{\Lambda}_{2,2} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and because the two matrices commute, i.e. $\mathbf{\Lambda}_{2,1}\mathbf{\Lambda}_{2,2} = \mathbf{\Lambda}_{2,2}\mathbf{\Lambda}_{2,1}$, then applying property (iv) of Lemma 2 we obtain

$$\mathbf{e}^{\mathbf{\Lambda_2 x}} = \mathbf{e}^{(\mathbf{\Lambda_{2,1}} + \mathbf{\Lambda_{2,2}})\mathbf{x}} = \mathbf{e}^{\mathbf{\Lambda_{2,1} x}}\,\mathbf{e}^{\mathbf{\Lambda_{2,2} x}}$$

where

$$\mathbf{e}^{\mathbf{\Lambda_{2,1} x}} = \begin{pmatrix} e^{\lambda x} & 0 \\ 0 & e^{\lambda x} \end{pmatrix} = e^{\lambda x}\mathbf{I}_2.$$

Using again formula (3.9) for matrix $\mathbf{\Lambda}_{2,2}$ we get

$$\mathbf{e}^{\mathbf{\Lambda_{2,2} x}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} + \frac{x^2}{2}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + ... = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

therefore multiplying by matrix $\mathbf{e}^{\boldsymbol{\Lambda}_{2,1}\mathbf{x}}$ yields (3.12).

In the third case, $\mathbf{\Lambda}_3$ is again non-diagonal, but it can also be decomposed into the sum of two matrices, $\mathbf{\Lambda}_{3,1}$ and $\mathbf{\Lambda}_{3,2}$, that commute

$$\mathbf{\Lambda}_3 = \mathbf{\Lambda}_{3,1} + \mathbf{\Lambda}_{3,2} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

Applying again property (iv) of Lemma 2 we get

$$\mathbf{e}^{\boldsymbol{\Lambda}_3\mathbf{x}} = \mathbf{e}^{\boldsymbol{\Lambda}_{3,1}\mathbf{x}}\,\mathbf{e}^{\boldsymbol{\Lambda}_{3,2}\mathbf{x}},$$

where

$$\mathbf{e}^{\boldsymbol{\Lambda}_{3,1}\mathbf{x}} = e^{\alpha x}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using again formula (3.9) for matrix $\mathbf{\Lambda}_{3,2}$ we get

$$\mathbf{e}^{\boldsymbol{\Lambda}_{3,2}\mathbf{x}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \beta\, x \\ -\beta\, x & 0 \end{pmatrix} + \frac{x^2}{2}\begin{pmatrix} \beta^2\, x^2 & 0 \\ 0 & -\beta^2\, x^2 \end{pmatrix} + \ldots = \begin{pmatrix} \cos\beta\, x & \sin\beta\, x \\ -\sin\beta\, x & \cos\beta\, x \end{pmatrix},$$

because $\sin y = \sum_{n=0}^{+\infty} \frac{y^{2n+1}}{(2n+1)}$ and $\cos y = \sum_{n=0}^{+\infty} \frac{y^{2n}}{(2n)}$, we obtain (3.13). □

For non-canonical matrices we have to specifically determine their exponential matrix:

Lemma 4 (Matrix exponential functions for non-canonical matrices)**.** *Let matrix* $\mathbf{A}$ *be in one of the two non-canonical forms, as in equation* (3.5)*. Then their matrices exponential functions are:*

1. *If* $\mathbf{A} = \mathbf{A}_d$*, then*

$$\mathbf{e}^{\mathbf{A_d}\mathbf{x}} = e^{\lambda x}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.14}$$

2. *if* $\mathbf{A} = \mathbf{A}_h$*, then* [4]

$$\mathbf{e}^{\mathbf{A}_h t} = e^{\alpha\, x}\begin{pmatrix} \cosh(\beta\, x) & \sinh(\beta\, x) \\ \sinh(\beta\, x) & \cosh(\beta\, x) \end{pmatrix} \tag{3.15}$$

Proof. We know that $\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$, where $\mathbf{\Lambda}$ is the Jordan form of $\mathbf{A}$. Then $\mathbf{e}^{\mathbf{A}\mathbf{x}} = \mathbf{e}^{\mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}\mathbf{x}} = \mathbf{P}\mathbf{e}^{\mathbf{x}}\mathbf{P}^{-1}$ by property (v) of Lemma 2. Matrix $\mathbf{A} = \mathbf{A}_d$ has two equal real eigenvalues equal to λ and, because it is diagonal it satisfies $\mathbf{A}_d\,\mathbf{P}_d = \mathbf{P}_d\,\mathbf{A}_d$. Therefore $\mathbf{P}_d = \mathbf{I}$ and

$$\mathbf{e}^{\mathbf{A}_d x} = \mathbf{P}\, e^{\lambda x}\,\mathbf{I}\,\mathbf{P}^{-1} = e^{\lambda x}\,\mathbf{I}.$$

Matrix $\mathbf{A} = \mathbf{A}_h$ has the real spectrum $\sigma = \{\ \alpha + \beta, \alpha - \beta\}$ and has eigenvector matrix

$$\mathbf{P}_h = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

[4]Recall $\cosh(\beta\, x) = \frac{1}{2}(e^{\beta x} + e^{-\beta x})$ and $\sinh(\beta\, x) = \frac{1}{2}(e^{\beta x} - e^{-\beta x})$

Therefore, the exponential matrix is

$$\mathbf{e}^{\mathbf{A}_h x} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{(\alpha+\beta)x} & 0 \\ 0 & e^{(\alpha+\beta)x} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

which, expanding the matrix multiplication, yields matrix (3.15). □

Summing up, the matrix exponential function can be reduced to two formal cases:

1. if matrix $\mathbf{A}$ is canonical, the matrix exponential is given by equation (3.10), which depends on the matrix exponential of its Jordan canonical form, which take one of following three forms (3.11), (3.12), or (3.13), depending on the spectrum of $\mathbf{A}$;
2. if matrix $\mathbf{A}$ is non-canonical, as in equation (3.5), its matrix exponential function is either given by equation (3.14) or (3.15).

3.4 The homogeneous planar ODE

In this section we present the general solution to the homogeneous linear planar ODE, that is to equation

$$\nabla \mathbf{y} = \mathbf{A}\,\mathbf{y},\ y : \mathrm{X} \subseteq \mathbb{R} \to \mathrm{Y} \subseteq \mathbb{R}^2. \tag{3.16}$$

Proposition 1 (General solution to the homogenous ODE (3.16))**.** *Consider the ODE* (3.16)*, for any real matrix* $\mathbf{A} \in \mathbb{R}^{2\times 2}$*. The unique solution is the mapping* $\mathbf{\Phi} : \mathrm{X} \times \mathrm{Y} \to \mathrm{Y} \subseteq \mathbb{R}^2$*,*

$$\mathbf{y}(x) = \mathbf{\Phi}(x, x_0, \mathbf{y}(x_0)) \equiv \mathbf{e}^{\mathbf{A}\,(x-x_0)}\,\mathbf{y}(x_0) \text{ for } x \geq x_0 \in \mathrm{X} \tag{3.17}$$

where $\mathbf{y}(x_0) \in \mathrm{Y}$ *is arbitrary.*

Proof. We can verify that the solution to equation (3.16) is (3.17). The derivative of (3.17) satisfies, from Lemma 2 (iii),

$$\frac{d}{dx}\mathbf{y}(x) = \frac{d}{dx}\mathbf{e}^{\mathbf{A}\,(x-x_0)}\mathbf{y}(x_0) = \mathbf{A}\,\mathbf{e}^{\mathbf{A}\,(x-x_0)}\mathbf{y}(x_0) = \mathbf{A}\,\mathbf{y}(x),$$

for any real matrix $\mathbf{A}$. □

We see that the solution is of the form $\mathbf{y}(x) = \Psi(x, x_0)\,\mathbf{y}(x_0)$ where

$$\Psi(x, x_0) = \mathbf{e}^{\mathbf{A}\,(x-x_0)}$$

is the **matrix exponential function** which encodes the dependence of the general solution of the ODE to the independent variable x.

Next we presents the several cases for matrix $\Psi(x, x_0)$, starting in subsection 3.4.1 with the cases in which $\mathbf{A}$ is in the canonical Jordan form or it is a non-canonical matrix, and continuing in subsection 3.4.2 with the general cases in which matrix $\mathbf{A}$ is not in the Jordan canonical form, but is similar to a Jordan canonical form.

We will see in the next chapter that the first cases contain the fundamental types of dynamic systems generated by planar linear ODE's.

3.4.1 A in a Jordan canonical form

Consider the ODE (3.16), such that $\mathbf{A} \in \{ \mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Lambda}_3\}$. From results in section 3.3 the matrix exponentials are

$$\Psi(x) \in \left\{ \begin{pmatrix} e^{\lambda_- x} & 0 \\ 0 & e^{\lambda_+ x} \end{pmatrix}, \begin{pmatrix} e^{\lambda x} & x \\ 0 & e^{\lambda x} \end{pmatrix}, e^{\alpha x} \begin{pmatrix} \cos\beta x & \sin\beta x \\ -\sin\beta x & \cos\beta x \end{pmatrix} \right\}. \tag{3.18}$$

If $\mathbf{A} \in \{ \mathbf{A}_d, \mathbf{A}_h\}$ the matrix exponentials are

$$\Psi(x) \in \left\{ \begin{pmatrix} e^{\lambda x} & 0 \\ 0 & e^{\lambda x} \end{pmatrix}, e^{\alpha x} \begin{pmatrix} \cosh\beta x & \sinh\beta x \\ \sinh\beta x & \cosh\beta x \end{pmatrix} \right\}. \tag{3.19}$$

We can gat more intuition if we expand equation (3.16), we have the following cases:

1. if $\mathbf{A} = \mathbf{\Lambda}_1$, the ODE takes the form

$$\begin{cases} y_1' = \lambda_- \, y_1, \\ y_2' = \lambda_+ \, y_2, \end{cases}$$

and has the solution

$$\mathbf{y}(x) = \begin{pmatrix} e^{\lambda_-(x-x_0)} & 0 \\ 0 & e^{\lambda_+(x-x_0)} \end{pmatrix} \mathbf{y}(x_0) = \begin{pmatrix} e^{\lambda_-(x-x_0)} \, y_1(x_0) \\ e^{\lambda_+(x-x_0)} y_2(x_0) \end{pmatrix} \tag{3.20}$$

2. if $\mathbf{A} = \mathbf{\Lambda}_2$, the ODE takes the form

$$\begin{cases} y_1' = \lambda \, y_1 + y_2, \\ y_2' = \lambda \, y_2 \end{cases}$$

and has the solution

$$\mathbf{y}(x) = \begin{pmatrix} e^{\lambda(x-x_0)} & x - x_0 \\ 0 & e^{\lambda(x-x_0)} \end{pmatrix} \mathbf{y}(x_0) = \begin{pmatrix} e^{\lambda(x-x_0)} \, y_1(x_0) + y_2(x_0) \, (x - x_0) \\ e^{\lambda(x-x_0)} \, y_2(x_0) \end{pmatrix} \tag{3.21}$$

3. if $\mathbf{A} = \mathbf{\Lambda}_3$, the ODE takes the form

$$\begin{cases} y_1' = \alpha \, y_1 + \beta \, y_2, \\ y_2' = -\beta \, y_1 + \alpha \, y_2; \end{cases}$$

and has the solution

$$\begin{aligned} \mathbf{y}(x) &= e^{\alpha(x-x_0)} \begin{pmatrix} \cos\beta(x-x_0) & \sin\beta(x-x_0) \\ -\sin\beta(x-x_0) & \cos\beta(x-x_0) \end{pmatrix} \mathbf{y}(x_0) \\ &= e^{\alpha(x-x_0)} \begin{pmatrix} y_1(x_0)\cos\beta(x-x_0) + y_2(x_0)\sin\beta(x-x_0) \\ -y_1(x_0)\sin\beta(x-x_0) + y_2(x_0)\cos\beta(x-x_0) \end{pmatrix}. \end{aligned} \tag{3.22}$$

The other two cases, i.e., if $\mathbf{A} = \mathbf{A}_d$ or $\mathbf{A} = \mathbf{A}_h$ have obvious solutions.

Observe that, while solution (3.21) correspond to a non-generic case, at it is relative to the case in which $\Delta(\mathbf{A}) = 0$, the other two cases are relative to both generic and non-generic cases. Therefore, we can have the following non-generic cases:

1. if $\mathbf{A} = \mathbf{\Lambda}_1$ and $\det(\mathbf{A}) = 0$,

$$\mathbf{y}(x) = \begin{pmatrix} y_1(x_0) \\ e^{\lambda_+(x-x_0)} y_2(x_0) \end{pmatrix}, \text{ if } \operatorname{trace}(\mathbf{A}) > 0, \text{ or } \mathbf{y}(x) = \begin{pmatrix} e^{\lambda_-(x-x_0)} y_1(x_0) \\ y_2(x_0) \end{pmatrix}, \text{ if } \operatorname{trace}(\mathbf{A}) < 0 \tag{3.23}$$

2. if $\mathbf{A} = \mathbf{\Lambda}_1$ and $\det(\mathbf{A}) = \operatorname{trace}(\mathbf{A}) = 0$,

$$\mathbf{y}(x) = \begin{pmatrix} y_1(x_0) \\ y_2(x_0) \end{pmatrix} \tag{3.24}$$

3. if $\mathbf{A} = \mathbf{\Lambda}_3$ and $\operatorname{trace}(\mathbf{A}) = 0$

$$\mathbf{y}(x) = \begin{pmatrix} y_1(x_0)\cos\beta(x-x_0) + y_2(x_0)\sin\beta(x-x_0) \\ -y_1(x_0)\sin\beta(x-x_0) + y_2(x_0)\cos\beta(x-x_0) \end{pmatrix}. \tag{3.25}$$

In the first two cases we observe that at least one element of $\mathbf{y}$ is constant, that is, depends only on the arbitrary element $x_0 \in \mathrm{X}$. in the second case the solutions trace out circular curves in Y, passing through a point $\mathbf{y}(x_0)$.

3.4.2 General A matrix

In this section we consider any (canonical) matrix $\mathbf{A}$, with the exception of cases $\mathbf{A}_d$ and $\mathbf{A}_h$, in equation (3.5). Equation (3.17) provides the general solution.

The superposition principle establishes a relationship between the solution of a ODE with a generic coefficient matrix $\mathbf{A}$, and an associated ODE whose coefficient matrix is the Jordan canonical form associated to $\mathbf{A}$, which we denote by $\mathbf{\Lambda}$.

Lemma 5 (Superposition principle). *Consider the coefficient matrix $\mathbf{A}$ and let $\mathbf{P}$ and $\mathbf{\Lambda}$ be its associated eigenvector matrix and Jordan canonical form. Then, then the solution of ODE* (3.35), *with general coefficient matrix $\mathbf{A}$, is*

$$\mathbf{y}(x) = \mathbf{P}\,\mathbf{w}(x), \text{ for any } x \in \mathrm{X} \tag{3.26}$$

where $\mathbf{w}$ is the solution of the ODE $\mathbf{w}' = \mathbf{\Lambda}\,\mathbf{w}$, that is $\mathbf{w}(x) = \Psi(x, x_0)\,\mathbf{w}(x_0)$ where $\Psi(x, x_0)$ is one of the matrices in equation (3.18) *and $\mathbf{w}(x_0) = \mathbf{P}^{-1}\ \mathbf{y}(x_0)$.*

Proof. Recall the transformation $\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$ where matrix $\mathbf{P}$ is non-singular. Then equation (3.37) yields $\mathbf{w}(x) = \mathbf{P}^{-1}\mathbf{y}(x)$. Taking derivatives for x we find $\mathbf{w}' = \frac{d\mathbf{w}}{dx} = \mathbf{P}^{-1}\,\mathbf{y}' = \mathbf{P}^{-1}\,\mathbf{A}\mathbf{y} = \mathbf{P}^{-1}\,\mathbf{A}\,\mathbf{P}\mathbf{y} = \mathbf{\Lambda}\mathbf{w}$. Equation $\mathbf{w}' = \mathbf{\Lambda}\,\mathbf{w}$ has solution $\mathbf{w}(x) = \Psi(x, x_0)\,\mathbf{w}(x_0)$, where $\Psi(x, x_0)$ is the form in (3.18) which is the matrix exponential for the Jordan form which is similar to $\mathbf{A}$. □

We call this transformation the superposition principle because the general solution to an ODE, with a general coefficient matrix, can be written as the sum of two general solutions. In the particular case in which matrix $\mathbf{A}$ has two real distinct eigenvalues, i.e., when $\Delta(\mathbf{A}) > 0$, the solution can be written as

$$\mathbf{y}(x) = \mathbf{P}^1\, w_1(x) + \mathbf{P}^2\, w_2(x)$$

where $\mathbf{P}^1$ and $\mathbf{P}^2$ are the eigenvectors of matrix $\mathbf{A}$.[5] This property is useful for characterizing the dynamics of the solution of an ODE when time is the independent variable.

An alternative form of the solution of a linear homogeneous ODE is

$$\mathbf{y}(x) = \mathbf{P}\, \Psi(x, x_0)\, \mathbf{P}^{-1}\;\; \mathbf{y}(x_0), \text{for any } \; x, x_0 \in \mathrm{X}$$

where $\Psi(x, x_0)$ is the matrix exponential associated to $\mathbf{A}$ which is given in equation (3.18).

3.5 Non-homogeneous equation

In this section we present solutions to the autonomous non-homogenous planar linear ODE

$$\nabla \mathbf{y} = \mathbf{A}\, \mathbf{y} + \mathbf{B}, \tag{3.27}$$

where $\mathbf{B}$ can be any real vector. In subsection 3.5.1 we assume that matrix $\mathbf{A}$ is in a Jordan canonical form, that is $\mathbf{A} = \mathbf{\Lambda}$ where $\mathbf{\Lambda} \in \{\, \mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Lambda}_3\}$, and in subsection 3.5.2 we consider an arbitrary coefficient matrix $\mathbf{A}$.

3.5.1 A in a Jordan canonical form

In this subsection we present the unique solutions of the planar linear ODE

$$\nabla \mathbf{y} = \mathbf{\Lambda} \mathbf{y} + \mathbf{B}. \tag{3.28}$$

It can take only one of the three expanded forms

$$\begin{cases} y_1' = \lambda_-\, y_1 + b_1, \\ y_2' = \lambda_+\, y_2 + b_2, \end{cases} \quad \begin{cases} y_1' = \lambda\, y_1 + y_2 + b_1, \\ y_2' = \lambda\, y_2 + b_2, \end{cases} \quad \text{and} \begin{cases} y_1' = \alpha\, y_1 + \beta\, y_2 + b_1, \\ y_2' = -\beta\, y_1 + \alpha\, y_2 + b_2 \end{cases} .$$

To study this equation it is useful to consider its **set of invariant solutions**, i.e., solutions in Y which are independent from $x \in n\mathrm{X}$,

$$\bar{\mathbf{y}} \; = \Big\{ \mathbf{y} \in \mathrm{Y} : \; \mathbf{\Lambda} \mathbf{y} + \mathbf{B} = \mathbf{0} \Big\}.$$

We show next that this set is non-empty, meaning invariant. solutions always exist, but it can contain several elements, meaning that invariant solutions may not be unique.

[5]Recall the the eigenvector matrix is the concatenation of the those two eigenvectors, $\mathbf{P} = \mathbf{P}^1 | \mathbf{P}^2$.

Lemma 6. *An invariant state always exists, and has the form*

$$\bar{\mathbf{y}} = -\mathbf{\Lambda}^{+}\,\mathbf{B} + \left(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\right)\mathbf{y} \tag{3.29}$$

where $\mathbf{\Lambda}^{+}$ *is the Moore-Penrose inverse of* $\mathbf{\Lambda}$ *and* $\mathbf{y}$ *is an arbitrary element of* Y*. If* $\det(\mathbf{\Lambda}) \neq 0$ *the invariant state is unique, and if* $\det(\mathbf{\Lambda}) = 0$ *there is an infinite number of invariant states.*

Proof. See (Magnus and Neudecker, 1988, p36). □

The following cases are possible.

Non-degenerate case If $\det(\mathbf{\Lambda}) \neq 0$ then the Moore-Penrose inverse is the classical inverse, that is

$$\mathbf{\Lambda}^{+} = \mathbf{\Lambda}^{-1} = \frac{\mathrm{adj}^{\top}(\mathbf{\Lambda})}{\det(\mathbf{\Lambda})},$$

where $\mathrm{adj}^{\top}(\mathbf{\Lambda})$ is the transposed of the adjoint matrix $\mathbf{\Lambda}$. The classic inverse satisfies the property $\mathbf{\Lambda}^{-1}\,\mathbf{\Lambda} = \mathbf{I}$. Then, the invariant state is unique, and from equation (3.29), it is

$$\bar{\mathbf{y}} = -\mathbf{\Lambda}^{-1}\,\mathbf{B}.$$

If $\mathbf{B} = \mathbf{0}$ then the invariant state is $\bar{\mathbf{y}} = \mathbf{0}$. In both cases, the invariant state is a **single point** in the set Y.

Degenerate cases If $\det(\mathbf{\Lambda}) = 0$ then $\Delta(\mathbf{\Lambda}) > 0$. Then all the eigenvalues are real, which means that the Jordan matrix $\mathbf{\Lambda}$ is diagonal, and it has at least one eigenvalue which is equal to zero. There is one zero eigenvalue if $\mathrm{trace}(\mathbf{A}) \neq 0$ and two zero eigenvalues if $\mathrm{trace}(\mathbf{A}) = 0$. This means that the Jordan matrix can only be one of the following three cases

$$\mathbf{\Lambda} \in \left\{ \begin{pmatrix} \lambda_{-} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \lambda_{+} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \tag{3.30}$$

The associated Moore-Penrose inverses are

$$\mathbf{\Lambda}^{+} \in \left\{ \begin{pmatrix} \frac{1}{\lambda_{-}} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\lambda_{+}} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \tag{3.31}$$

Therefore, substituting those matrices in equation (3.29) we find

$$\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda} \in \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and there is always an infinite number of invariant states depending on the arbitrary element $\mathbf{y} \in \mathrm{Y}$. It is useful to consider further two possibilities: first, if $\mathrm{trace}(\mathbf{A}) \neq 0$ from equation (3.29), we find the invariant states are

$$\bar{\mathbf{y}} = \begin{pmatrix} -\frac{b_1}{\lambda_{-}} \\ y_2 \end{pmatrix}, \text{ or } \bar{\mathbf{y}} = \begin{pmatrix} y_1 \\ -\frac{b_2}{\lambda_{+}} \end{pmatrix}. \tag{3.32}$$

In both cases the set of invariant states defines a **one-dimensional linear manifold** (i.e, a line) in the two-dimensional set Y: in the first case it is the line defined by $y_1 = -\frac{b_1}{\lambda_-}$ (a vertical line in a Cartesian diagram), and in the second it is the line defined by $y_2 = -\frac{b_2}{\lambda_+}$ (a horizontal line in a Cartesian diagram); and, second, if trace($\mathbf{\Lambda}$) = 0 there is also an infinite number of invariant states

$$\bar{\mathbf{y}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathrm{Y}, \tag{3.33}$$

which means that the set of invariant states is coincident with set Y, i.e., $\bar{\mathbf{y}} = \mathrm{Y}$, which we can see as a **two-dimensional manifold** (a surface).

Corollary 1. *An invariant state always exists. Furthermore, it is unique if* $\det(\mathbf{\Lambda}) \neq 0$*, and there is an infinite number if* $\det(\mathbf{\Lambda}) = 0$.

Next, we obtain a general form for the solution of ODE (3.28), for any matrices $\mathbf{\Lambda}$ and $\mathbf{B}$.

Proposition 2 (General solution to the non-homogenous ODE (3.28))**.** *Consider the ODE* (3.28) *for an arbitrary real vector* $\mathbf{B} \in \mathbb{R}^2$*. The solution to the ODE always exists and is uniquely given by*

$$\mathbf{y}(x) = -\mathbf{\Lambda}^+ \ \mathbf{B} + \ \mathbf{e}^{\mathbf{\Lambda}\,(x-x_0)}\,(\mathbf{y}(x_0) + \mathbf{\Lambda}^+ \ \mathbf{B}) + (\mathbf{I} - \mathbf{\Lambda}^+\,\mathbf{\Lambda})\,\mathbf{B}\,(x-x_0), \ \textit{for}\ x, x_0 \in \mathrm{X} \tag{3.34}$$

where $\mathbf{e}^{\mathbf{\Lambda}\,(x-x_0)}$ *is the appropriate matrix exponential given in equation* (3.18)*,* $\mathbf{y}(x_0)$ *is an arbitrary element of* Y*, associated to an arbitrary* $x_0 \in \mathrm{X}$.

Proof. We start with the case in which $\det(\mathbf{\Lambda}) \neq 0$. Then again, matrix $\mathbf{\Lambda}$ has a unique classical inverse, $\mathbf{\Lambda}^+ = \mathbf{\Lambda}^{-1}$, which implies that $\bar{\mathbf{y}} = -\mathbf{\Lambda}^{-1}\,\mathbf{B}$ and $\mathbf{I} - \mathbf{\Lambda}^+\mathbf{\Lambda} = \mathbf{0}$. Define $\mathbf{z}(x) = \mathbf{y}(x) - \bar{\mathbf{y}}$ where $\mathbf{y}$ is given in equation (3.29). Then $\nabla\mathbf{z} = \nabla\mathbf{y} = \mathbf{\Lambda}\,\mathbf{y} + \mathbf{B} = \mathbf{\Lambda}\,(\mathbf{y} - \bar{\mathbf{y}}) = \mathbf{\Lambda}\,\mathbf{z}$, yields a homogenous ODE $\nabla\mathbf{z} = \mathbf{\Lambda}\,\mathbf{z}$, whose solution is, from equation (3.17), $\mathbf{z}(x) = e^{\mathbf{\Lambda}(x-x_0)}\mathbf{z}(x_0)$. Going back to the original variables we have

$$\mathbf{y}(x) = -\mathbf{\Lambda}^{-1}\,\mathbf{B} + e^{\mathbf{\Lambda}(x-x_0)}\,(\mathbf{y}(x_0) + \mathbf{\Lambda}^{-1}\,\mathbf{B})$$

If $\det(\mathbf{A}) = 0$ the coefficient matrix takes one of the forms in equation (3.30). Therefore, the ODE's can take one of the following forms

$$\begin{cases} y_1' = \lambda_-\,y_1 + b_1, \\ y_2' = b_2, \end{cases} \quad \begin{cases} y_1' = b_1, \\ y_2' = \lambda_+\,y_2 + b_2, \end{cases} \quad \text{or} \quad \begin{cases} y_1' = b_1, \\ y_2' = b_2. \end{cases}$$

Using the results for the scalar ODE, the solutions are

$$\begin{cases} y_1(x) = -\frac{b_1}{\lambda_-} + e^{\lambda_-(x-x_0)}\left(y_1(x_0) + \frac{b_1}{\lambda_-}\right) \\ y_2(x) = y_2(x_0) + b_2\,(x-x_0) \end{cases} \quad \begin{cases} y_1(x) = y_1(x_0) + b_1\,(x-x_0) \\ y_2(x) = -\frac{b_2}{\lambda_+} + e^{\lambda_+(x-x_0)}\left(y_2(x_0) + \frac{b_2}{\lambda_+}\right) \end{cases} \quad \text{or}$$

$$\begin{cases} y_1(x) = y_1(x_0) + b_1\,(x-x_0) \\ y_2(x) = y_2(x_0) + b_2(x-x_0). \end{cases}$$

If we consider that: first, the invariant states in the first and second cases are the same we obtained in equation (3.32) , for the first two cases, and (3.33) for the third case; second, the exponential matrices are, respectively

$$\begin{pmatrix} e^{\lambda_- x} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda_+ x} \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

third, their Jordan matrices in equation (3.30); and, at last, their Moore-Penrose inverses iin equation (3.31), then we see that equation (3.34) is the matrix form encompassing all cases. □

If $\det(\mathbf{A}) \neq 0$ then the solution can be written as

$$\mathbf{y}(x) = \bar{\mathbf{y}} + \Psi(x, x_0)\left(\mathbf{y}(x_0) - \bar{\mathbf{y}}\right), \; x, x_0 \in \mathrm{X}$$

where $\bar{\mathbf{y}} = -\mathbf{\Lambda}^{-1} \; \mathbf{B}$ is the unique invariant state, and $\Psi(x, x_0) = e^{\mathbf{\Lambda}(x-x_0)}$ is the matrix exponential.

3.5.2 Generic A matrix

In this section we solve the general planar ODE

$$\nabla \mathbf{y} = \mathbf{A}\mathbf{y} + \mathbf{B} \tag{3.35}$$

where matrix $\mathbf{A}$ is not necessarily in a Jordan canonical form and $\mathbf{B}$ can be any real vector. This covers both the homogenous case in which $\mathbf{B} = \mathbf{0}$ and the non-homogeneous case in which $\mathbf{B} \neq \mathbf{0}$.

Proposition 3 (Invariant state for the non-homogenous ODE (3.35))**.** *invariant states for equation* (3.35) *exist and are given by*

$$\bar{\mathbf{y}} = -\mathbf{A}^+ \mathbf{B} + \left(\mathbf{I} - \mathbf{A}^+ \mathbf{A}\right)\mathbf{y}, \tag{3.36}$$

where $\mathbf{A}^+ = \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}$ *is the Moore-Penrose inverse of* $\mathbf{A}$*, and* $\mathbf{y}$ *is an arbitrary element of* Y*.*

Proof. Multiplying equation (3.37) by $\mathbf{P}$ we get

$$\begin{aligned}
\bar{\mathbf{y}} &= \mathbf{P}\,\bar{\mathbf{w}} \\
&= -\mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}\,\mathbf{B} + \mathbf{P}(\mathbf{I} - \mathbf{\Lambda}^+\,\mathbf{\Lambda})\,\mathbf{w}(0) \\
&= -A^+\,\mathbf{B} + \mathbf{P}(\mathbf{I} - \mathbf{\Lambda}^+\,\mathbf{\Lambda})\,\mathbf{P}^{-1}\;\mathbf{y}(0) \\
&= -A^+\mathbf{B} + (\mathbf{P}\,\mathbf{P}^{-1} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1}\;)\,\mathbf{y}(0) \\
&= -A^+\mathbf{B} + (\mathbf{I} - \mathbf{A}^+\,\mathbf{P}\,\mathbf{P}^{-1}\;\;\mathbf{A})\,\mathbf{y}(0) \\
&= -A^+\mathbf{B} + (\mathbf{I} - \mathbf{A}^+\;\;\mathbf{A})\,\mathbf{y}(0)
\end{aligned}$$

□

In order to find the solution of the ODE (3.35), we start by presenting two useful results:

Lemma 7. *Consider the coefficient matrix* $\mathbf{A}$ *and let* $\mathbf{P}$ *and* $\mathbf{\Lambda}$ *be its associated eigenvector matrix and Jordan canonical form. Then, the ODE* (3.35) *with general coefficient matrix* $\mathbf{A}$ *can be transformed into an ODE with coefficient matrix* $\mathbf{\Lambda}$

$$\mathbf{y}(x) = \mathbf{P}\,\mathbf{w}(x) \tag{3.37}$$

where $\mathbf{P}$ *is the eigenvector matrix associated to* $\mathbf{A}$ *and* $\mathbf{w}(x)$ *is the solution of the ODE*

$$\nabla\mathbf{w} = \mathbf{\Lambda}\,\mathbf{w} + \mathbf{P}^{-1}\,\mathbf{B} \tag{3.38}$$

Proof. Recall that any matrix satisfies $\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$ where matrix $\mathbf{P}$ is non-singular. Then we can introduce a unique linear transformation $\mathbf{w}(x) = \mathbf{P}^{-1}\mathbf{y}(x)$. Then

$$\nabla\mathbf{w} = \mathbf{P}^{-1}\,\nabla\mathbf{y} = \mathbf{P}^{-1}\left(\mathbf{A}\mathbf{y} + \mathbf{B}\right) = \mathbf{\Lambda}\mathbf{P}^{-1}\mathbf{y} + \mathbf{P}^{-1}\mathbf{B} = \mathbf{\Lambda}\mathbf{w} + \mathbf{P}^{-1}\mathbf{B}.$$

□

Lemma 8. *The solution to the ODE transformed coordinates* $\mathbf{w}$*, equation* (3.38) *is*

$$\mathbf{w}(x) = \bar{\mathbf{w}} \; + \mathbf{e}^{\mathbf{\Lambda}(x-x_0)}(\mathbf{w}(x_0) - \bar{\mathbf{w}}) + (\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda})\,\mathbf{P}^{-1}\,\mathbf{B}\,(x - x_0) \tag{3.39}$$

where

$$\bar{\mathbf{w}} \; = -\mathbf{\Lambda}^{+}\,\mathbf{P}^{-1}\,\mathbf{B} + (\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda})\,\mathbf{w}(0)$$

and $\mathbf{w}(x_0) = \mathbf{P}^{-1}\,\mathbf{y}(x_0)$ *for an arbitrary* $\mathbf{y}(x_0)$*.*

Proof. ODE (3.38) is a non-homogeneous ODE in which the coefficient matrix is in the Jordan canonical form. Comparing with equation (3.28) we find that instead of $\mathbf{B}$ we now have $\mathbf{P}^{-1}\mathbf{B}$. By performing this substitution in the solution to the last ODE, in equation (3.34) we find the solution of the transformed ODE in equation (3.39). □

The general solution to equation (3.35) exists and is uniquely given in the next proposition:

Proposition 4 (Solution for the non-homogenous ODE (3.35))**.** *Consider the ODE* (3.35) *for any matrix* $\mathbf{A} \in \mathbb{R}^{2\times 2}$ *and vector* $\mathbf{B} \in \mathbb{R}^{2}$*. The solution to the ODE always exist and is uniquely given by*

$$\mathbf{y}(x) = \bar{\mathbf{y}} + \; \mathbf{e}^{\mathbf{A}\,(x-x_0)}\,(\mathbf{y}(x_0) - \bar{\mathbf{y}}) + (\mathbf{I} - \mathbf{A}^{+}\,\mathbf{A})\,\mathbf{B}\,(x - x_0),\; \textit{for}\, x, x_0 \in \mathrm{X}, \tag{3.40}$$

where the invariant state $\bar{\mathbf{y}}$ *is given in equation* (3.36)*, and* $\mathbf{y}(x_0)$ *is an arbitrary element of* $\mathbf{y}$ *for* $x = x_0$*.*

Proof. Multiplying equation (3.37) by $\mathbf{P}$ we get the inverse transformation $\mathbf{y}(x) = \mathbf{P}\,\mathbf{w}(x)$. Using the solution for the transformed variables in equation (3.39) yields

$$\begin{aligned} y(x) &= \mathbf{P}\,\bar{\mathbf{w}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda}(x-x_0)}(\mathbf{w}(0) - \bar{\mathbf{w}}) + \mathbf{P}(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda})\,\mathbf{P}^{-1}\,\mathbf{B}\,(x - x_0) \\ &= \bar{\mathbf{y}} \; + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda}(x-x_0)}\mathbf{P}^{-1}(\mathbf{y}(x_0) - \bar{\mathbf{y}}) + \mathbf{P}(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda})\,\mathbf{P}^{-1}\,\mathbf{B}\,(x - x_0) \\ &= \bar{\mathbf{y}} + e^{\mathbf{A}(x-x_0)}(\mathbf{y}(0) - \bar{\mathbf{y}}) + (\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}\;)\,\mathbf{B}\,(x - x_0) \end{aligned}$$

which gives equation (3.40). □

Next we present the specific forms for the ODE (3.35).

Solutions for $\det(\mathbf{A}) \neq 0$ cases

If $\det(\mathbf{A}) \neq 0$ then $\mathbf{A}^+ = \mathbf{A}^{-1}$ then there is a unique invariant state

$$\bar{y} = -\mathbf{A}^{-1}\,\mathbf{B}.$$

Expanding the previous formula, we have

$$\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = -\frac{1}{\det(\mathbf{A})} \begin{pmatrix} a_{22}\,b_1 - a_{12}\,b_2 \\ -a_{21}\,b_1 - a_{11}\,b_2 \end{pmatrix}.$$

The solution (3.40) takes the particular form

$$\mathbf{y}(x) = \bar{\mathbf{y}} \; + \mathbf{e}^{\mathbf{A}(x-x_0)}\,(\mathbf{y}(x_0) - \bar{\mathbf{y}}), \tag{3.41}$$

where $\mathbf{e}^{\mathbf{A}t} = \mathbf{P}\,\mathbf{e}^{\mathbf{\Lambda}t}\,\mathbf{P}^{-1}$, where $\mathbf{e}^{\mathbf{\Lambda}t}$ is the matrix exponential of the Jordan canonical form which is similar to $\mathbf{A}$. It is useful in applications to write the solution (3.41) as

$$\mathbf{y}(x) = \overline{\mathbf{y}} + \mathbf{P}\,\mathbf{e}^{\mathbf{\Lambda}x}\;\mathbf{k}(x_0)$$

where $\mathbf{k}(x_0) = \mathbf{P}^{-1}(\mathbf{y}(x_0) - \bar{\mathbf{y}})$. Writing the eigenvector matrix $\mathbf{P}$ as [6]

$$\mathbf{P} = \mathbf{P}^-|\mathbf{P}^+ \; = \begin{pmatrix} P_1^- & P_1^+ \\ P_2^- & P_2^+ \end{pmatrix},$$

then

$$\mathbf{k}(x_0) = \begin{pmatrix} k_1(x_0) \\ k_2(x_0) \end{pmatrix} = \frac{1}{\det(\mathbf{P})} \begin{pmatrix} P_2^+ & -P_1^+ \\ -P_2^- & P_1^- \end{pmatrix} \begin{pmatrix} y_1(x_0) - \overline{y}_1 \\ y_2(x_0) - \overline{y}_2 \end{pmatrix}.$$

in which $\mathbf{y}(x_0)$ is an arbitrary element of $\mathbf{y}$ for $x = x_0$.

Then the solution for the non-degenerate cases can take the following forms

1. If $\Delta(\mathbf{A}) > 0$ then the Jordan canonical form of matrix $\mathbf{A}$ is $\mathbf{\Lambda}_1$ and the general solution is

$$\mathbf{y}(x) = \overline{\mathbf{y}} + k_1(x_0)\,\mathbf{P}^-\,e^{\lambda_-(x-x_0)} + k_2(x_0)\,\mathbf{P}^+\,e^{\lambda_+(x-x_0)}$$

   where $\mathbf{P}^-$ ($\mathbf{P}^+$) is the simple eigenvector associated with λ_- (λ_+). More specifically

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + k_1(x_0) \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} e^{\lambda_-(x-x_0)} + k_2(x_0) \begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix} e^{\lambda_+(x-x_0)}. \tag{3.42}$$

2. If $\Delta(\mathbf{A}) = 0$ then the Jordan canonical form of matrix $\mathbf{A}$ is $\mathbf{\Lambda}_2$. The general solution is

$$\mathbf{y}(x) = \overline{\mathbf{y}} + e^{\lambda(x-x_0)}\left(\mathbf{P}^1(k_1(x_0) + k_2(x_0)\,(x-x_0)) + k_2(x_0)\,\mathbf{P}^2\right)$$

   where $\mathbf{P}^1$ is a simple eigenvector and $\mathbf{P}^2$ is a generalized eigenvector (see the Appendix), or, equivalently

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + e^{\lambda(x-x_0)}\left((k_1(x_0) + k_2(x_0)\,(x-x_0)) \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} + k_2(x_0) \begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix} \right)$$

[6] Recall that $\mathbf{P}^j$ is the solution of the homogeneous system $(\mathbf{A} - \lambda_j\mathbf{I})\,\mathbf{P}^j = \mathbf{0}$.

3. If $\Delta(\mathbf{A}) < 0$ then the Jordan canonical form of matrix $\mathbf{A}$ is $\mathbf{\Lambda}_3$. The general solution is

$$\mathbf{y}(x) = \overline{\mathbf{y}} + e^{\alpha(x-x_0)}\Big((k_1(x_0)\cos\beta(x-x_0) + k_2(x_0)\,\sin\beta(x-x_0))\mathbf{P}^1 + \\ + (k_2(x_0)\,\cos\beta(x-x_0) - k_1(x_0)\,\sin\beta(x-x_0))\mathbf{P}^2\Big).$$

where $\mathbf{P}$ is a eigenvector (see the Appendix for the determination of the eigenvector matrix in the case in which the eigenvectors are complex) or, equivalently,

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + e^{\alpha(x-x_0)}\Bigg\{\ k_1(x_0)\begin{pmatrix} P_1^1\cos\beta(x-x_0) - P_1^2\,\sin\beta(x-x_0) \\ P_2^1\cos\beta(x-x_0) - P_2^2\sin\beta(x-x_0) \end{pmatrix} + \\ +\ k_2(x_0)\begin{pmatrix} P_1^1\sin\beta(x-x_0) + P_1^2\,\cos\beta(x-x_0) \\ P_2^1\sin\beta(x-x_0) + P_2^2\cos\beta(x-x_0) \end{pmatrix}\Bigg\}.$$

Solutions for $\det(\mathbf{A}) = 0$ **cases**

Degenerate cases occur for $\det(\mathbf{A}) = 0$ implying that $\mathbf{A}^+ \neq \mathbf{A}^{-1}$ and that the Jordan canonical form is diagonal (i.e, of type $\mathbf{\Lambda}_1$ in which one or two of the eigenvalues are equal to zero).

As $\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$ then $\mathbf{A}^+ = \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}$ and $\mathbf{A}^+\,\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}\,\mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1} = \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$ where $\mathbf{\Lambda}$ is one of the Jordan forms in equation (3.30) and $\mathbf{\Lambda}^+$ is the associated the Moore-Penrose in equation (3.31), depending on the trace being $\text{trace}(\mathbf{A}) \neq 0$ or $\text{trace}(\mathbf{A}) = 0$.

First observe that (3.40) can be expanded as

$$\mathbf{y}(x) = -\mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}\,\mathbf{B} + \ \mathbf{e}^{\mathbf{A}\,(x-x_0)}\,(\mathbf{y}(0) + \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}\,\mathbf{B}) + (\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1})\,\mathbf{B}\,(x-x_0),$$

where we can see that there are some components which are independent from the particular Jordan form in equation (3.30) and others which depend on the particular Jordan form.

For the first case we have $\tilde{\mathbf{B}} = \mathbf{P}^{-1}\,\mathbf{B}$ and $\mathbf{w}(0) = \mathbf{P}^{-1}\,\mathbf{y}(0)$, and write their expansion as

$$\tilde{\mathbf{B}} = \begin{pmatrix} \tilde{b}_- \\ \tilde{b}_+ \end{pmatrix} = \frac{1}{\det(\mathbf{P})}\begin{pmatrix} -P_2^-\,b_1 + P_1^-\,b_2 \\ P_2^+\,b_1 - P_1^+\,b_2 \end{pmatrix}$$

and

$$\mathbf{w}(x_0) = \begin{pmatrix} w_-(x_0) \\ w_+(x_0) \end{pmatrix} = \frac{1}{\det(\mathbf{P})}\begin{pmatrix} -P_2^-\,y_1(x_0) + P_1^-\,y_2(x_0) \\ P_2^+\,y_1(x_0) - P_1^+\,y_2(x_0) \end{pmatrix}$$

For the second case we have, if $\lambda_- < 0 = \lambda_+$,

$$\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1} = \frac{1}{\det(\mathbf{P})}\begin{pmatrix} -P_2^-\,P_1^+ & P_1^-\,P_1^+ \\ -P_2^-\,P_2^+ & P_1^-\,P_2^+ \end{pmatrix}$$

for the case in which $\lambda_- = 0 < \lambda_+$ we have

$$\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1} = \frac{1}{\det(\mathbf{P})}\begin{pmatrix} P_2^+\,P_1^- & -P_1^+\,P_1^- \\ P_2^+\,P_2^- & -P_1^+\,P_2^- \end{pmatrix}$$

and for $\lambda_- = \lambda_+ = 0$ we have $\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1} = \mathbf{I}$.

Therefore the solutions become

1. if $\lambda_- < 0 = \lambda_+$

$$\mathbf{y}(x) = \mathbf{P}^+ w_+(x_0) - \mathbf{P}^- \frac{\tilde{b}_-}{\lambda_-} + \begin{pmatrix} P_1^- \, e^{\lambda_-(x-x_0)} \\ P_2^- \end{pmatrix} \, (w_-(x_0) + \frac{\tilde{b}_-}{\lambda_-}) - \mathbf{P}^+ \tilde{b}_+$$

2. if $\lambda_- = 0 < \lambda_+$

$$\mathbf{y}(x) = \mathbf{P}^- w_-(x_0) - \mathbf{P}^+ \frac{\tilde{b}_+}{\lambda_+} + \begin{pmatrix} P_1^+ \\ P_2^+ \, e^{\lambda_+(x-x_0)} \end{pmatrix} \, (w_+(x_0) + \frac{\tilde{b}_+}{\lambda_+}) - \mathbf{P}^- \tilde{b}_-$$

3. for $\lambda_- = \lambda_+ = 0$

$$\mathbf{y}(x) = \mathbf{P} \left(\mathbf{w}(x_0) - \tilde{b}_- \right).$$

3.6 Applications

3.6.1 A firm's allocation of sales across locations

Consider the problem a firm located at $x = 0$, producing the quantity μ, wants to distributed in a circular area of given length L, $\mathrm{X} = \left[-\frac{L}{2}, \frac{L}{2} \right]$. The firm is a price taker has faces two types of costs when trying to moving output: it has a quadratic investment cost to be able to set up a network of stores, etc; and second has a transport cost which is proportional to the output to be displaced. Denoting by $y(x)$ the sales at location x and by $u(x)$ the investment cost to be able to delivery to location x, and it the firm wants to maximize the profit, the firm's problem becomes:

$$\begin{aligned} \max_{u(\cdot)} \; & \int_{-L/2}^{L/2} \; p \, y(x) - \frac{1}{2} \left(u(x) \right)^2 dx \\ & \text{subject to} \\ & y'(x) = u(x) - \delta \, y(x) \\ & \int_{-L/2}^{L/2} y(x) \, dx = \mu \\ & y(-\tfrac{L}{2}) = y(\tfrac{L}{2}) \end{aligned} \tag{3.43}$$

The optimality conditions are (see chapter on optimal control)

$$\begin{cases} y'(x) = u(x) - \delta \, y(x) & \text{for } \; x \in \mathrm{X} \\ u'(x) = u(x) - p, & \text{for } \; x \in \mathrm{X} \\ \int_{-L/2}^{L/2} y(x) \, dx = \mu, & \text{for } \; x \in \mathrm{X} \\ y(-\frac{L}{2}) = y(\frac{L}{2}) & \text{for } \; x \in \{ -\frac{L}{2}, \frac{L}{2} \; \}. \end{cases}$$

Next, we want to use the previous theory to finding the solution to this problem. First, we write the planar differential equation in matrix form

$$\begin{pmatrix} y'(x) \\ u'(x) \end{pmatrix} = \begin{pmatrix} -\delta & 1 \\ 0 & \delta \end{pmatrix} + \begin{pmatrix} 0 \\ -p \end{pmatrix}$$

The coefficient matrix has det $= -\delta^2$ and therefore is non-singular. Its Jordan canonical form and eigenvector matrix is

$$\mathbf{\Lambda} = \begin{pmatrix} -\delta & 0 \\ 0 & \delta \end{pmatrix}, \ \mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 2\,\delta \end{pmatrix}$$

Furthermore, the invariant state is

$$\begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} = -\mathbf{A}^{-1}\ \mathbf{B} = \frac{p}{\delta^2}\ \begin{pmatrix} 1 \\ \delta \end{pmatrix}$$

Taking $x_0 = 0$, the general solution for the planar differential equations is, using (3.42) because our ODE is non-homogeneous and non-singular

$$\begin{pmatrix} y(x) \\ u(x) \end{pmatrix} = \frac{p}{\delta^2}\ \begin{pmatrix} 1 \\ \delta \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\delta\,x} \ + k_2 \begin{pmatrix} 1 \\ 2\,\delta \end{pmatrix} e^{\delta\,x}$$

where k_1 and k_2 are arbitrary constants.

To find the particular solution, i.e., to determine the values for the arbitrary constants k_1 and k_2 we use the two side-conditions, which only involve the variable $y(\cdot)$

Using the first side-condition we find

$$\int_{-L/2}^{L/2} y(x)\,dx = \mu \iff \frac{p\,L}{\delta^2} + \frac{k_1}{\delta}\left(e^{\delta L/2} - \ e^{-\delta L/2}\right) + \frac{k_2}{\delta}\left(e^{\delta L/2} - \ e^{-\delta L/2}\right) = \mu.$$

and, using the second side-condition yiedls

$$y(-\tfrac{L}{2}) = y(\tfrac{L}{2}) \iff k_1 = k_2.$$

Therefore,

$$k_1 = k_2 = \frac{\mu\delta^2 - p\,L}{2\,\delta}\left(e^{\delta\,L/2} - e^{-\delta\,L/2}\right)^{-1}.$$

Them optimal distribution of output accross region X is

$$y^*(x) = \frac{p}{\delta^2} \ + \frac{\mu\delta^2 - p\,L}{2\,\delta}\left(\frac{e^{-\delta\,x} \ + e^{\delta\,x}}{e^{\delta\,L/2} - e^{-\delta\,L/2}}\right) \tag{3.44}$$

which is illustrated in Figure 3.2 for particular values of the parameters.

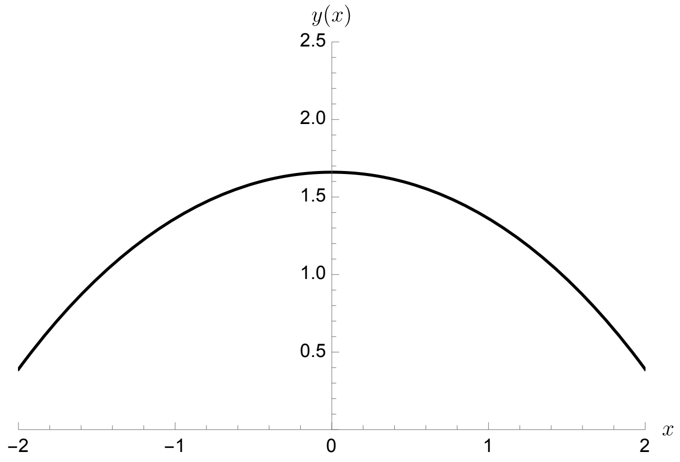


Figure 3.2: Solution, in equation (3.44) for $p = 1$, $\delta = 0.5$, $L = 4$ and $\mu = 5$.

3.7 References

Mathematics: Perko (1996)

3.A Appendix

3.A.1 Review of matrix algebra

Consider matrix $\mathbf{A}$ of order 2 with real entries

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

that is $\mathbf{A} \in \mathbb{R}^{2\times 2}$. The **trace** and the **determinanx** of $\mathbf{A}$ are, respectively,

$$\text{trace}(\mathbf{A}) = a_{11} + a_{22},\ \det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}.$$

The kernel (or null space) of matrix $\mathbf{A}$ is a vector $\mathbf{v}$ defined as

$$\text{kern}(\mathbf{A}) = \{\ \mathbf{v} : \mathbf{A}\mathbf{v} = \mathbf{0}\}$$

The dimension of the kernel gives a measure of the linear independence between the rows of $\mathbf{A}$.

The characteristic polynomial of matrix $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda\mathbf{I}_2) = \lambda^2 - \text{trace}(\mathbf{A})\lambda + \det(\mathbf{A}) \tag{3.45}$$

where $\lambda \in \mathbb{C}$ is an eigenvalue, which is complex valued.

The **spectrum** of $\mathbf{A}$ is the **set of eigenvalues**

$$\sigma(\mathbf{A}) \equiv \{\ \lambda \in \mathbb{C} : \det(\mathbf{A} - \lambda\mathbf{I}_2)\ = 0\}$$

The **eigenvalues** of any 2×2 matrix $\mathbf{A}$ are

$$\lambda_+ = \frac{\text{trace}(\mathbf{A})}{2} + \Delta(\mathbf{A})^{\frac{1}{2}},\ \lambda_- = \frac{\text{trace}(\mathbf{A})}{2} - \Delta(\mathbf{A})^{\frac{1}{2}} \tag{3.46}$$

where the discriminant is

$$\Delta(\mathbf{A}) \equiv \left(\frac{\text{trace}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A}).$$

A useful result on the relationship between the eigenvalues and the trace and the determinant of $\mathbf{A}$:

Lemma 9. *Let λ_+ and λ_- be the eigenvalues of a 2×2 matrix $\mathbf{A}$. Then they are verify:*

$$\begin{aligned} \lambda_+ + \lambda_- &= \text{trace}(\mathbf{A}) \\ \lambda_+\lambda_- &= \det(\mathbf{A}). \end{aligned}$$

Three cases can occur:

1. if $\Delta(\mathbf{A}) > 0$ then λ_+ and λ_- are real and distinct and $\lambda_+ > \lambda_-$
2. if $\Delta(\mathbf{A}) = 0$ then $\lambda_+ = \lambda_- = \lambda = \text{trace}(\mathbf{A})/2$ are real and multiple,

3. if $\Delta(\mathbf{A}) < 0$ then λ_+ and λ_- are complex conjugate $\lambda_+ = \alpha + \beta i$ and $\lambda_- = \alpha - \beta i$ where $\alpha = \frac{\text{tr}(A)}{2}$ and $\beta = \sqrt{|\Delta(\mathbf{A})|}$ and $i = \sqrt{-1}$.

In the last case, we can write the eigenvalues in polar coordinates as

$$\lambda_+ = r(\cos\theta + \sin\theta i),\ \lambda_- = r(\cos\theta - \sin\theta i)$$

where $r = \sqrt{\alpha^2 + \beta^2}$ and $\tan\theta = \beta/\alpha$, or

$$\alpha = r\cos\theta,\ \beta = r\sin\theta$$

Jordan canonical forms Two matrices $\mathbf{A}$ and $\mathbf{A}'$ with the equal eigenvalues are called **similar**. This allows for classifying matrices according to their eigenvalues.

The Jordan canonical forms for 2×2 matrices are

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix},\ \mathbf{\Lambda}_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},\ \mathbf{\Lambda}_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{3.47}$$

Lemma 10 (Jordan canonical from of matrix $\mathbf{A}$)**.** *Consider any 2×2 matrix with real entries and its discriminant $\Delta(\mathbf{A})$. Then*

1. If $\Delta(\mathbf{A}) > 0$ then the Jordan canonical form associated to $\mathbf{A}$ is $\mathbf{\Lambda}_1$.

2. If $\Delta(\mathbf{A}) = 0$ then the Jordan canonical form associated to $\mathbf{A}$ is $\mathbf{\Lambda}_2$.

3. If $\Delta(\mathbf{A}) < 0$ then the Jordan canonical form associated to $\mathbf{A}$ is $\mathbf{\Lambda}_3$.

The Jordan canonical form $\mathbf{\Lambda}_3$ can also be represented by a diagonal matrix with complex entries

$$\mathbf{\Lambda}_3 = \begin{pmatrix} \alpha + \beta i & 0 \\ 0 & \alpha - \beta i \end{pmatrix}.$$

In this sense, if $\Delta(\mathbf{A}) \neq 0$ then matrix $\mathbf{A}$ is diagonalizable and it is not diagonalizable if $\Delta(\mathbf{A}) = 0$. Figure 3.1 presents the different cases in a $(\text{trace}(\mathbf{A}), \det(\mathbf{A}))$ diagram.

It has the following information:

- Jordan canonical forms are associated to the following areas: $\mathbf{\Lambda}_1$ is outside the parabola; $\mathbf{\Lambda}_3$ is inside the parabola, and $\mathbf{\Lambda}_2$ is represented by the parabola;
- in the positive orthant the two eigenvalues have positive real parts, in the negative orthant they have negative real parts and bellow the abcissa there are two real eigenvalues with opposite signs;
- the abcissa corresponds to the locus of points in which there is at least one zero-valued eigenvalue, the upper part of the ordinate corresponds to complex eigenvalues with zero real part, and the origin to the case in which there are two eigenvalues equal to zero.

Eigenvectors of A

Lemma 11. *Let* $\mathbf{A}$ *be a* 2×2 *matrix with real entries. Then, there exists a non-singular matrix* $\mathbf{P}$ *such that*

$$\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$$

where $\mathbf{\Lambda}$ *is the Jordan canonical form of* $\mathbf{A}$*, and matrix* $\mathbf{P}$ *is a* 2×2 ***eigenvector*** *matrix associated to* $\mathbf{A}$*.*

There are two types of eigenvectors:

1. **simple eigenvectors** if $\Delta(\mathbf{A}) \neq 0$. In this case the eigenvector is $\mathbf{P} = (\mathbf{P}^{-}, \mathbf{P}^{+})$ concatenating the eigenvectors $\mathbf{P}^{-}$ and $\mathbf{P}^{+}$ associated to the eigenvalues λ_{+} and λ_{-}, which are obtained from solving the homogeneous system

$$(\mathbf{A} - \lambda_j \mathbf{I}_2)\mathbf{P}^j = 0, \; j = 1, 2$$

   where $\mathbf{I}_2$ is the identity matrix of order 2. Observe that $\mathbf{P}^j = \text{kern}(\mathbf{A} - \lambda_j \mathbf{I}_2)$, i.e, it is the null space of matrix $(\mathbf{A} - \lambda_j \mathbf{I}_2)$;

2. **generalized eigenvectors** if $\Delta(\mathbf{A}) = 0$, that is, when we have multiple eigenvalues $\lambda_{+} = \lambda_{-} = \lambda$. In this case we determine $\mathbf{P} = (\mathbf{P}^1, \mathbf{P}^2)$ where $\mathbf{P}^1$ is a simple eigenvalue and $\mathbf{P}^2$ is a generalized eigenvalue. They are obtained in the following way: first, $\mathbf{P}^1$ solves $(\mathbf{A} - \lambda \mathbf{I})\mathbf{P}^1 = 0$, where $\mathbf{I} = \mathbf{I}_2$; second, (a) if $(\mathbf{A} - \lambda \mathbf{I})^2 \neq \mathbf{0}$ we determine $\mathbf{P}^2$ from $(\mathbf{A} - \lambda \mathbf{I})^2 \mathbf{P}^2 = 0$; however, (b) if $(\mathbf{A} - \lambda \mathbf{I})^2 = \mathbf{0}$ then we determine $\mathbf{P}^2$ from $(\mathbf{A} - \lambda \mathbf{I})\mathbf{P}^2 = \mathbf{P}^1$.

When $\Delta(\mathbf{A}) < 0$ we can use one of the following two approaches:

1. either we write the Jordan matrix as a complex-valued matrix

$$\Lambda_3 = \begin{pmatrix} \alpha + \beta i & 0 \\ 0 & \alpha - \beta i \end{pmatrix}$$

   and compute $\mathbf{P}^j$ as a complex-valued vector from

$$(\mathbf{A} - \lambda_j \mathbf{I}_2)\mathbf{P}^j = 0,$$

2. or we write the Jordan matrix as a real-valued matrix as in equation (3.47) and compute $\mathbf{P}$ as a real-valued matrix by setting $\mathbf{P} = (\mathbf{u}, \mathbf{v})$ where $\mathbf{Q} = \mathbf{u} + \mathbf{v}i$ is the solution of the homogeneous system

$$(\mathbf{A} - (\alpha + \beta i)\mathbf{I}_2)\mathbf{Q} = 0$$

Conclusion: given a matrix $\mathbf{A}$, we can find matrices $\mathbf{\Lambda}$ and $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\Lambda\mathbf{P}^{-1}$ where $\mathbf{P}$ is invertible. Equivalently $\Lambda = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

Proposition 5. *The eigenvector matrices associated to the Jordan canonical forms are:*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{3.48}$$

for $\Lambda = \mathbf{\Lambda}_1$, $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ *and* $\mathbf{\Lambda} = \mathbf{\Lambda}_3$, *respectively*

Proof. For $\mathbf{\Lambda} = \mathbf{\Lambda}_1$, because $(\mathbf{\Lambda}_1 - \lambda_+ \mathbf{I})\mathbf{P}^- = 0$ and $(\mathbf{\Lambda}_1 - \lambda\_\mathbf{I})\mathbf{P}^+ = 0$ are

$$\begin{pmatrix} 0 & 0 \\ 0 & \lambda_- - \lambda_+ \end{pmatrix} \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda_+ - \lambda_- & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then we get $\mathbf{P} = (\mathbf{P}^- \mathbf{P}^+) = \mathbf{I}$, because $\lambda_+ \neq \lambda_-$. For $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ we determine the simple eigenvector from $(\mathbf{\Lambda}_2 - \lambda \mathbf{I})\mathbf{P}^- = 0$. To determine the second eigenvector as $(\lambda\_ - \lambda \mathbf{I})^2 = \mathbf{0}$, because

$$(\lambda\_ - \lambda \mathbf{I})^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then we use $(\mathbf{\Lambda}_2 - \lambda \mathbf{I})\mathbf{P}^2 = \mathbf{P}^1$,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix} = \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix},$$

to get $\mathbf{P}^1 = (1, 0)$ and $\mathbf{P}^2 = (1, 1)$.

For $\mathbf{\Lambda} = \mathbf{\Lambda}_3$ consider eigenvalue $\lambda_+ = \alpha + \beta i$ and assume that there is a complex vector

$$\mathbf{z} = \begin{pmatrix} u_1 + v_1 i \\ u_2 + v_2 i \end{pmatrix}$$

that solves $(\Lambda_3 - (\alpha + \beta i)I)\mathbf{z} = 0$, that is [7]

$$\begin{cases} \beta\,(u_2 + v_1 + (v_2 - u_1)i) & = 0 \\ \beta\,((v_2 - u_1) - (u_2 + v_1)i) & = 0 \end{cases}$$

then we should have $u_1 = v_2$ and $u_2 = -v_1$. We can arbitrarily set $u_1 = 1$ and $v_1 = 1$, in $\mathbf{P}^1 = (u_1, u_2)^\top$ and $\mathbf{P}_2 = (v_1, v_2)^\top$, to get the third eigenvector matrix. □

Eigenspaces As matrix $\mathbf{P}$ is non singular it forms a basis for vector space $\mathbf{A}$. Then vector space $\mathbf{A}$ can be seen as a direct sum $\mathbf{A} = \mathcal{E}^1 \oplus \mathcal{E}^2$ where

$$\begin{aligned} \mathcal{E}^1 &= \{\text{eigenspace associated with } \lambda_+\} \\ \mathcal{E}^2 &= \{\text{ eigenspace associated with } \lambda\_\}. \end{aligned}$$

[7] We use the rules for sums and multiplications of complex numbers: if $x_1 = a_1 + b_1 i$ and $x_2 = a_2 + b_2 i$, then $x_1 + x_2 = (a_1 + a_2) + (b_1 + b_2)i$ and $x_1 x_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$ because $i^2 = -1$.

Chapter 4

Planar linear autonomous ODE dynamics

4.1 Introduction

This chapter considers planar linear autonomous ordinary differential equations having time as the independent variable. In th eprevious chapter, we have already presented complete analytical, or explicit, solutions to this equation for a generic independent variable. In this chapter we present a qualitative (or geometric) characterization of its solution. This also constitutes an introduction to the ODE approach to dynamic systems, which has an important branch in applied mathematics and is crucial for the understanding of economic dynamics.

Knowledge of the dynamics of the linear planar ODE is not only interesting *per se*, but also because, from the Grobman-Hartmann theorem (see chapter on non-linear ODEs), it provides conditions under which the dynamics of non-linear ODEs can be (at least locally) qualitatively characterized from the properties of an associated linear ODE.

A large proportion of dynamic systems in economics are either linear or have a dynamics which is topologically equivalent to a linear ODE. In particular, we will see that a thorough characterizations of the solution to optimal control problems, which cannot be obtained explicitly in most models, can be achieved by linearization, i.e., by approximating unknown solutions by solutions provided by an equivalent linear ODE.

Planar ODEs feature some new types of dynamics, when compared to the scalar case: first, although asymptotic stability and (global) instability cases can exist, as in the scalar case, the existence of saddle point dynamics (or conditional stability) is a new type of dynamics for the planar case; second in addition to monotonic solution paths, as in the scalar case, several types of non-monotonic solution paths can exist in the planar case. The saddle-point case is a very common type of dynamics in both macroeconomics and growth theory and charaterizes solutions of most optimal control problems.

The general (autonomous) **linear planar ordinary differential equation**, is a linear functional equation over the two-dimensional variable over the set $\mathrm{T} \subseteq \mathbb{R}_+$, $\mathbf{y} : \mathrm{T} \to \mathbb{R}^2$, and its derivative, $\dot{y} : \mathrm{T} \to \mathbb{R}^2$, where $\mathrm{T} = [t_0, \infty)$, usually $t_0 = 0$, where

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \; \dot{\mathbf{y}}(t) \equiv \begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix},$$

In explicit matrix form, the ODE is

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}, \; \mathbf{y} : \mathrm{T} \subseteq \mathbb{R}_+ \to \mathbf{y} \subseteq \mathbb{R}^2. \tag{4.1}$$

where $\mathbf{A} \in \mathbb{R}^{2\times 2}$, is areal values matrix, and $\mathbf{B} \in \mathbb{R}^{2\times 1}$ is a real valued vector

$$\mathbf{A} \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; \mathbf{B} \equiv \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{4.2}$$

We showed in chapter 3 that an explicit solution to the ODE always exists, and can take one of the two following general forms. First. if if $\det(\mathbf{A}) \neq 0$ there is an unique stationary (i.e., time-independent solution) $\bar{\mathbf{y}} = -\mathbf{A}^{-1}\,\mathbf{B}$, if $\bar{\mathbf{y}} \in \mathrm{Y}$, such that

$$\mathbf{y}(t) = \mathbf{\Phi}(t; t_0, \mathbf{y}(t_0); \mathbf{A}, \mathbf{B}) \equiv \bar{\mathbf{y}} + \mathbf{e}^{\mathbf{A}(t-t_0)}\,(\mathbf{y}(t_0) - \bar{y}), \tag{4.3}$$

where t_0 is an arbitrarily fixed point in time and $\mathbf{y}(t_0) \in \mathrm{Y}$ is an arbitrary value associated with $t = t_0$ Second, if $\det(\mathbf{A}) = 0$ the solution is

$$\mathbf{y}(t) = \mathbf{\Phi}(t; t_0, \mathbf{y}(t_0); \mathbf{A}, \mathbf{B}) \equiv \bar{\mathbf{y}} + \mathbf{e}^{\mathbf{A}(t-t_0)}\,(\mathbf{y}(t_0) - \bar{y}) + (\mathbf{I} - \mathbf{A}^+\,\mathbf{A})\mathbf{B}(t - t_0) \tag{4.4}$$

where $\bar{\mathbf{y}} = -\mathbf{A}^+\,\mathbf{B} + (\mathbf{I} - \mathbf{A}^+\,\mathbf{A})\mathbf{y}(t_0)$ and $\mathbf{A}^+ = \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}$, where $\mathbf{\Lambda}^+$ is the Moore-Penrose inverse of the Jordan canonical form associated with $\mathbf{A}$. In this case, there is either an infinite number of steady states or a steady state does not exist.

Equations (4.3) or (4.4) are also called **general solution** to the ODE. They trace out a family of trajectories $\left(\mathbf{y}(t)\right)_{\mathrm{T}}$, which have three main features that will concern us in the rest of this chapter. First, the type of of its behavior over time, which is detremined by the the algebraic properties of matrix $\mathbf{A}$. Second, the location, and sometimes the existence, of steady states, which also depends on vector $\mathbf{B}$. At last, its consistency and dependence from the side-conditions regarding the pair $(t, \mathbf{y}(t)) = (0, \mathbf{y}(0))$, which should be introduced when we specify a model, or a problem which encompassses the ODE.

As we saw, a solution to the ODE always exists and are unique, and solutions to problems involving ODEs always exist but may not be unique.

For scalar ODE's, we saw that for going from general solutions to particular solutions, which are completely specified functions, we have to introduce one side condition. When time is an independent variable, the side condition took the form of an initial or a terminal condition. For planar ODE's obtaining **particular solutions**, or completely specified solutions, we need to introduce **two** side conditions. If the two side conditions involve known values at time $t_0 = 0$, as $\mathbf{y}(t_0) = \mathbf{y}_0$, we say we have an **initial-value problem**, if there is one side condition for the initial value and another for the terminal (if T is finite) or asymptotic (if $T \to \infty$) the problem can be called **mixed-value problem**, and if the two conditions are on the terminal or asymptotic state we can call it **terminal-value problem**.[1] The previous ODE's can be called forward, backward, and forward-backward equations, respectively.

In most of this chapter we consider instead equation (4.1) in the **expanded form**

$$\begin{aligned} \dot{y}_1 &= a_{11}y_1 + a_{12}y_2 + b_1 \\ \dot{y}_2 &= a_{21}y_1 + a_{22}y_2 + b_2. \end{aligned} \tag{4.5}$$

This chapter proceeds as follows. In section 4.2 we present the geometrical approach to solving linear planar ODEs. In section 4.3 the algebraic solutions of the ODE is characterized from the eigenvalues of matrix $\mathbf{A}$, section 4.4 is a brief introduction to bifurcation analysis, section 4.5 shows how to transform second order scalar ODEs into a linear planar ODE. The last two sections present the main types of ODE problems, in section 4.6, and the main structures of problem in economics, in section 4.7.

4.2 The geometry of planar ODE's

The **geometric** approach for solving ODE consists in drawing a **phase diagram**. As for scalar ODE's a geometrical representation by a phase diagram is a way characterizing the qualitative properties of the solution in the space Y. Indeed is a way of "solving" the ODE equation without performing algebraic or numerical computations.

Figure 4.1 and table 4.1 present all possible phase diagrams for a planar linear autonomous ODE, which is called a bifurcation diagram, whose detailed derivation is one of the purposes of this chapter. A bifurcation diagram represents all the possible phase diagrams which an ODE can have, depending on the values of its parameters.

A phase diagram for planar autonomous ODE is a geometrical representation of the dynamics in the two-dimensional space $Y \subseteq \mathbb{R}^2$. It contains the following elements:

1. **isoclines (or nullclines)** are the geometrical loci in the space Y such that one of the variables y_1 or y_2 is constant. There are two isoclines, the first associated to y_1 and the second associated with y_2

   $$\mathbb{I}_{y_1} = \{\, \mathbf{y} \in Y : \dot{y}_1 = 0\}, \text{ and } \mathbb{I}_{y_2} = \{\, \mathbf{y} \in Y : \dot{y}_2 = 0\}.$$

   Looking at equation (4.5) it should be evident that every isocline is a line in the space Y. The steady states are the locus or loci where isoclines intersect or are coincident;

[1] If the independent variable is not time the last two cases are usually called **boundary-value problems**.

Table 4.1: Stability of steady states

	$\det(\mathbf{A}) < 0$	$\det(\mathbf{A}) = 0$	$\det(\mathbf{A}) > 0$		
			$\Delta(\mathbf{A}) < 0$	$\Delta(\mathbf{A}) = 0$	$\Delta(\mathbf{A}) > 0$
$\text{trace}(\mathbf{A}) < 0$	saddle	stable saddle-node	stable focus	stable node with multiplicity	stable node
$\text{trace}(\mathbf{A}) = 0$	saddle	degenerate saddle-node	center		
$\text{trace}(\mathbf{A}) > 0$	saddle	unstable saddle-node	unstable focus	unstable node with multiplicity	unstable node

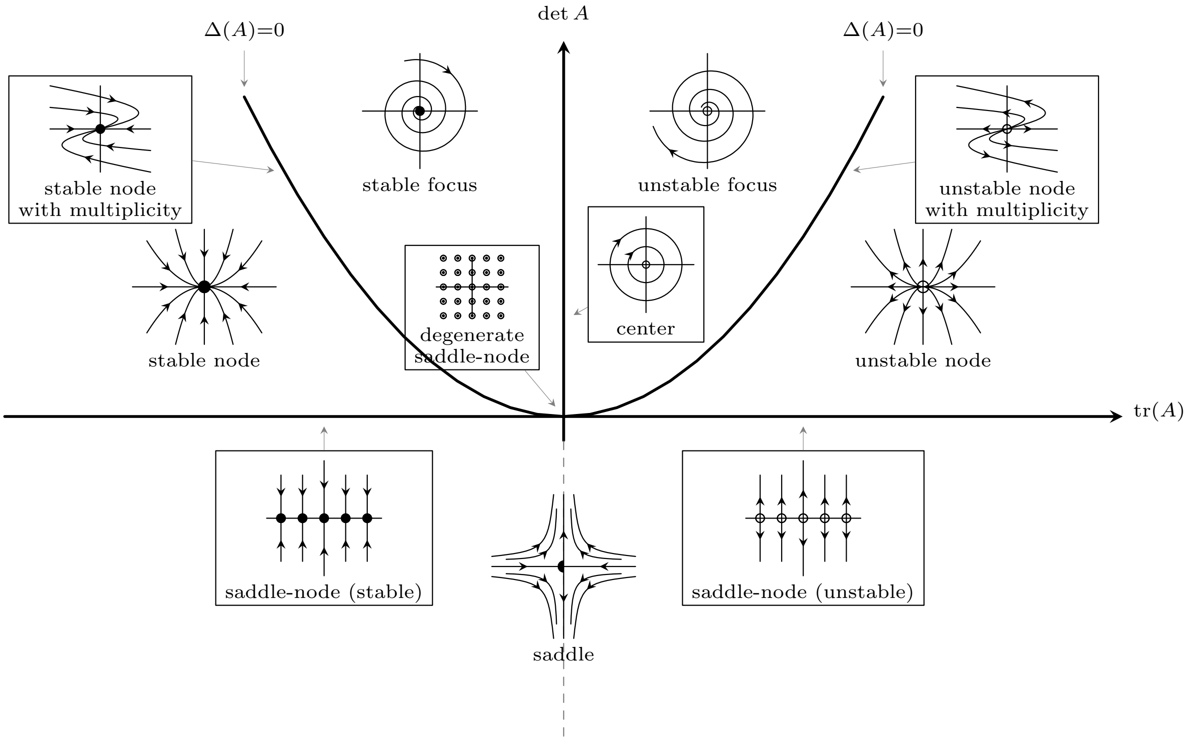


Figure 4.1: Bifurcation diagram in the $(\text{trace}A, \det A)$ space

2. the two isoclines divide the set Y into four **quadrants**

$$\begin{aligned} \mathrm{Y}^{++} &= \{\, \mathbf{y} \in \mathrm{Y} : \dot{y}_1 > 0,\ \dot{y}_2 > 0\} \\ \mathrm{Y}^{-+} &= \{\, \mathbf{y} \in \mathrm{Y} : \dot{y}_1 < 0,\ \dot{y}_2 > 0\} \\ \mathrm{Y}^{+-} &= \{\, \mathbf{y} \in \mathrm{Y} : \dot{y}_1 > 0,\ \dot{y}_2 < 0\} \\ \mathrm{Y}^{--} &= \{\, \mathbf{y} \in \mathrm{Y} : \dot{y}_1 < 0,\ \dot{y}_2 < 0\} \end{aligned},$$

   where each quadrant represent a particular joint change in time for both variables (increasing if $\dot{y}_j > 0$ and decreasing if $\dot{y}_j < 0$). This allows us to represent the direction of the forward evolution of both variables in a grid of points in Y;

3. the **vector field** represents the resultant of those two directions, for every point, which indicates the direction of evolution of the solution $\mathbf{y}(t)$;

4. the **eigenspaces** $\mathcal{E}^-$ and $\mathcal{E}^+$ are lines in $\mathbf{y}$ whose slopes are given by those of the eigenvectors $\mathbf{P}^-$ and $\mathbf{P}^+$. Their representation allows us to have a geometric representation of the stable, unstable and center manifolds, $\mathcal{W}^s$, $\mathcal{W}^u$, and $\mathcal{W}^c$, which are lines or two-dimensional subsets of Y. These subsets introduce another partition to set Y associated to the stability of the solution, that is, to its convergence towards the steady state (when we are dealing with a forward ODE);

5. some representative trajectories, usually starting from points $\mathbf{y}(0)$ located in each one of the four quadrants, which are called **integral curves**. They are parametric curves representing the solution to the ODE within space Y, in which time is implicit. In order to take account of the direction of the movement, they are usually represented with direction arrows showing the direction of the solution over time.

4.2.1 Normal forms for planar linear ODE's

Recalling results from chapter 3, we say a planar linear autonomous ODE's is in a **normal form** if it is of the form

$$\dot{\mathbf{y}} = \mathbf{A}\,\mathbf{y} + \mathbf{B}.$$

where, using the results from chapter 3, we assume $\mathbf{A}$ is in one of the three Jordan canonical forms or it is equal to one of the two non-canonical matrices, that is $\mathbf{A} = \{\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Lambda}_3, \mathbf{\Lambda}_d, \mathbf{\Lambda}_h\}$.

Associated to those matrices we have the following types of phase diagrams

- if $\mathbf{A} = \mathbf{\Lambda}_1$, and matrix $\mathbf{A}$ has two real and distinct eigenvalues we have a **node** if both eigenvalues have the same sign and are non-zero, a **saddle** if they are real and have different signs, or a **saddle-node** if there one zero eigenvalue;
- if $\mathbf{A} = \mathbf{\Lambda}_2$ matrix $\mathbf{A}$ has two equal real eigenvalues and we have a **node with multiplicity**;
- if $\mathbf{A} = \mathbf{\Lambda}_3$ matrix $\mathbf{A}$ has two complex conjugate eigenvalues we have a **focus**.

Figure 4.1 illustrates the four main types of phase diagrams that exist for a planar linear ODE, which are determined by the algebraic properties of matrix $\mathbf{A}$: **nodes**, if all eigenvalues are real and do not have symmetric signs, **saddles**, if all eigenvalues are real and have symmetric signs, **foci** if the two eigenvalues are complex conjugate with non-zero real parts, and **centers** if the two eigenvalues are complex conjugate with zero real parts.

Next we present the main phase diagrams. We detail in the first case the construction of a saddle, which is the most common phase diagram in economics. In the other phase diagrams we point out the main differences.

4.2.2 Building a phase diagram

We show with a simple example two points: how to build the phase diagram and the reason of calling the previous ODE normal forms.

We show with four examples how to represent geometrically a saddle, in particular how the phase diagram changes when with $\mathbf{A}$ in a Jordan canonical form or in a similar matrix, and for homogeneous or non-homogeneous sytems.

We start with the simplest case in which matrix $\mathbf{A}$ is a diagonal matrix, that is $\mathbf{A} = \mathbf{\Lambda}_1$, for a homogeneous equation, i.e., for $\mathbf{B} = 0$.

Example 1 Consider the planar linear homogenous ODE where $\mathbf{y} \in \mathrm{Y} = \mathbb{R}^2$.

$$\dot{y}_1 = -3y_1$$
$$\dot{y}_2 = 3y_2.$$

The coefficient matrix $\mathbf{A}$ is the Jordan form $\mathbf{\Lambda}_1$

$$\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}.$$

We study the geometry of the solution by building the phase diagram in Figure 4.2. The next steps are followed:

First, we build panel (a) by considering equation $\dot{y}_1 = -3\,y_1$. As $\dot{y}_1 = 0$ if $y_1 = 0$, and $\dot{y}_1 < 0$ ($\dot{y}_1 > 0$) if $y_1 > 0$ ($y_1 < 0$), then, for every value of y_2, the isocline for the first equation $\{\ \mathbf{y} : y_1 = 0\}$ is the vertical axis. This isocline partitions set Y by separating the subset of Y, for which y_1 increases, from the subset of Y, for which y_1 decreases, as the horizontal arrows show.

Second,we build panel (b) by considering equation $\dot{y}_2 = 3\,y_2$. As $\dot{y}_2 = 0$ if $y_2 = 0$, and $\dot{y}_2 < 0$ ($\dot{y}_2 > 0$) if $y_2 < 0$ ($y_2 > 0$), then, for every value of y_1, the isocline for the second equation $\{\mathbf{y} : y_2 = 0\}$ is the horizontal axis. This isocline partitions set Y by separating the subset of Y, for which y_2 increases, from the subset of Y, for which y_2 decreases, as the vertical arrows show.

Third, panel (c) superimposes the two previous diagrams, and shows the four quadrants we have referred. It provides several insights on the dynamics of the ODE: (1) the two isoclines intersect at a steady state, and because they intersect only once we conclude that the steady state exists, it is unique, and in this case it is the origin (i.e., $\overline{\mathbf{y}}\ = (0,0)$; (2) by depicting the resultant of the arrows traced out in panels (a) and (b), passing to representative points in the diagram, we have a geometric representation of the vector field.

At last, panel (d) shows the phase diagram, which includes some representative trajectories where the ODE is interpreted as a forward ODE. It shpws that initial points located along the horizontal axis converge to the steady state, which means that the stable manifold coincides with the horizontal axis, i.e, to points $\mathbf{y} = (y_1, 0)$ for arbitrary y_1; and that any initial point not belonging to the horizontal axis will generate a flow that will converge to the vertical axis such that $\lim_{t\to\infty}\ \mathbf{y}(t) = (0, \pm\infty)$.

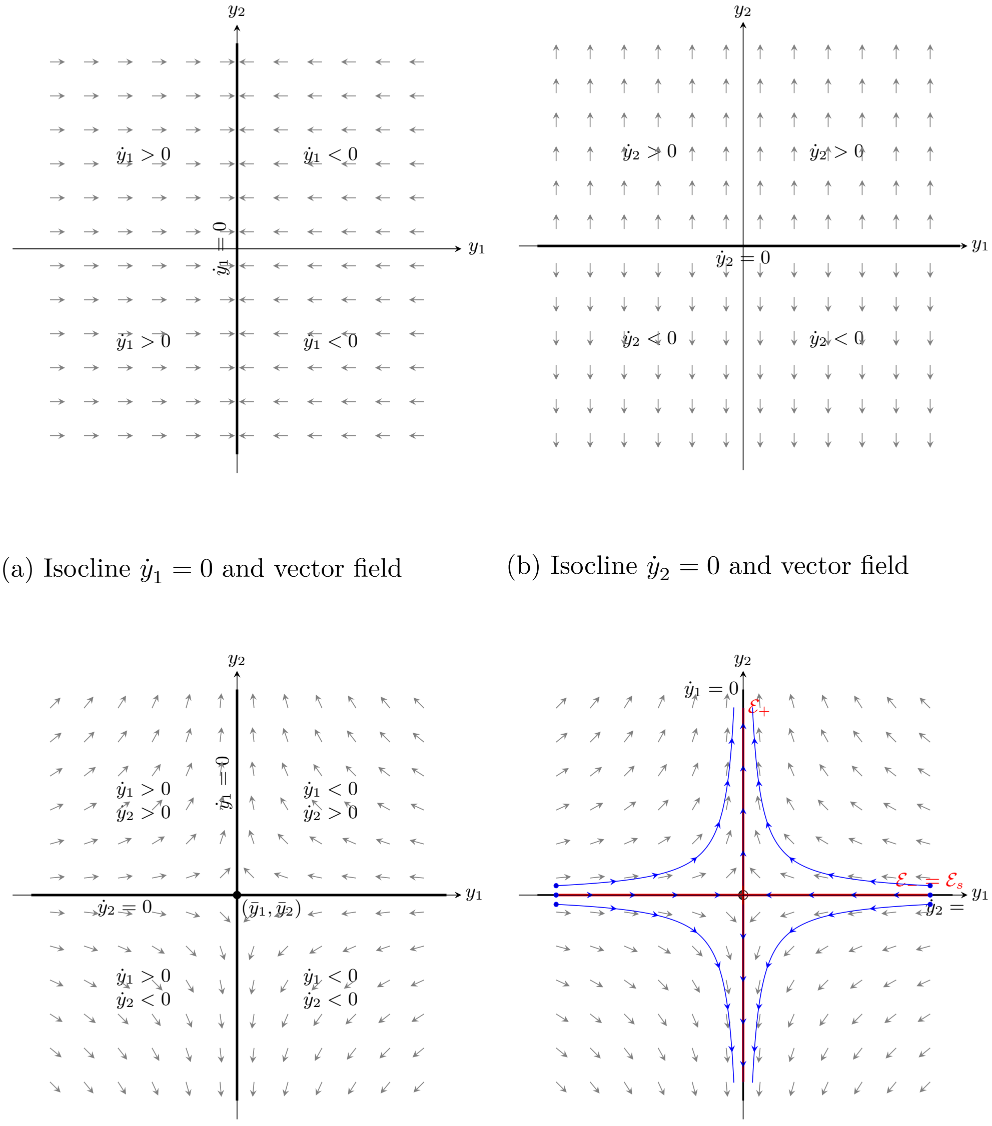
This geometric intuition is confirmed by the analytical solution of the ODE. We can use the algebraic approach presented in this and in chapter 3.

Summing up, we shows that: (1) there is a unique steady state $\overline{\mathbf{y}}\ = (\overline{y}_1, \overline{y}_2) = (0,0)$; (2) as trace$(\mathbf{A})\ = 0$ and det$\,(\mathbf{A})\ = -9$, the eigenvalues of the coefficient matrix are $\lambda_- = -3$ and $\lambda_+ = 3$, and, therefore, the steady state is a saddle point; (3) the associated eigenvectors are $\mathbf{P}^1 = (1,0)^\top$ and $\mathbf{P}^2 = (0,1)^\top$; (4) this implies that the eigenspaces associated to the eigenvalues λ_- and λ_+ are

$$\mathcal{E}^- = \{\ \mathbf{y} \in \mathbb{R}^2 : y_2 = 0\},\ \mathcal{E}^+ = \{\ \mathbf{y} \in \mathbb{R}^2 : y_1 = 0\};$$

and, therefore, the center manifold $\mathcal{W}^c$ is empty and the stable and unstable manifolds are both of dimension 1 and are

$$\mathcal{W}^s = \mathcal{E}^-,\ \mathcal{W}^u = \mathbb{R}^2/\mathcal{W}^s$$



(a) Isocline $\dot{y}_1 = 0$ and vector field

(b) Isocline $\dot{y}_2 = 0$ and vector field

(c) The four quadrants, and the vector field

(d) Phase diagram

Figure 4.2: Example 1: Building the phase diagram.

meaning that for any $\mathbf{y} \neq (y_1(0), 0)$ the solution is unstable.

Furthermore, the (general) solution of the ODE is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y_1(0)\, e^{-3t} \\ y_2(0)\, e^{3t} \end{pmatrix} = y_1(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t} + y_2(0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t},\ t \in [0, \infty).$$

There are two types of particular solutions paths, $\left(\mathbf{y}(t)\right)_{t \in \mathrm{T}}$, exist if we observe that $\lim_{t\to\infty}\ e^{3t}\ = +\infty$ and $\lim_{t\to\infty}\ e^{-3t}\ = 0$:

- if $\mathbf{y}(0) \in \mathcal{E}^-$, that is $y_2(0) = 0$ the solutions are

$$\begin{pmatrix} y_1^s(t) \\ y_2^s(t) \end{pmatrix} = \begin{pmatrix} y_1(0)e^{-3t} \\ 0 \end{pmatrix}, \text{ for } t \in [0, \infty).$$

  They converge asymptotically to the steady state, that is

$$\lim_{t\to\infty} \begin{pmatrix} y_1^s(t) \\ y_2^s(t) \end{pmatrix} = \lim_{t\to\infty} \begin{pmatrix} y_1(0)e^{-3t} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  These are trajectories belonging to $\mathcal{W}^s$. In economics these trajectories are commonly called saddle paths and are omnipresent in DGE models.

- if $\mathbf{y}(0) \notin \mathcal{E}^-$, that is, if $y_2(0) \neq 0$, the solutions are unbounded

$$\lim_{t\to\infty} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \lim_{t\to\infty} y_2(0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} 0 \\ \pm\infty \end{pmatrix}$$

  These are trajectories belonging to $\mathcal{W}^u$ which converge over time to the unstable manifold $\mathcal{E}^+$.

All these algebraic results confirm the qualitative intuition we obtained from drawing the phase diagram, which is not only a geometrical representation of the ODE but also a powerful way to obtain a fast intuition on the dynamics of a planar ODE.

Example 2 Consider the planar linear non-homogenous ODE where $\mathbf{y} \in \mathrm{Y} = \mathbb{R}^2$,

$$\begin{aligned} \dot{y}_1 &= -3y_1 + 1 \\ \dot{y}_2 &= 3y_2 - 1, \end{aligned}$$

where the coefficient matrix is equal to the one in Example 1, and the vector $\mathbf{B} \neq \mathbf{0}$,

$$\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}, \text{and } \mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Figure 4.3 panel (a) shows the phase diagram. Comparing to example Example 1 (see Figure 4.2 panel (d)) we see that the two isoclines are shifted but keep the same slopes. This entails: first, moving the steady state to the positive orthant away from the origin; second, the isocline have the same slopes but $\dot{y}_1 = 0$ is shifted to the right, to $y_1 = \frac{1}{3}$ and $\dot{y}_2 = 0$ is shifted up to $y_2 = \frac{1}{3}$; and, at last, the stable and unstable manifolds, $\mathcal{E}^-$ and $\mathcal{E}^+$, are still coincident with the isoclines $\dot{y}_2 = 0$ and $\dot{y}_1 = 0$, respectively. The stable manifold is $\mathcal{W}^s = \left\{ \mathbf{y} : y_2 = \frac{1}{3} \right\}$.

Therefore, now the steady state is shifted from $\overline{\mathbf{y}} = (0, 0)$ to $\overline{\mathbf{y}} = (\frac{1}{3}, \frac{1}{3})$, and the general solution to the ODE is now

$$\begin{aligned} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} &= \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + \begin{pmatrix} \left(y_1(0) - \frac{1}{3}\right) e^{-3t} \\ \left(y_2(0) - \frac{1}{3}\right) e^{3t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + \left(y_1(0) - \frac{1}{3}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t} + \left(y_2(0) - \frac{1}{3}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t}, \ t \in [0, \infty). \end{aligned}$$

Again, there are two main types of trajectories:

- if $\mathbf{y}(0) \in \mathcal{E}^-$, that is, if $y_2(0) = \frac{1}{3}$, the trajectories $\left(\mathbf{y}(t)\right)_{t\in \mathrm{T}}$ belong to the stable manifold, $\mathcal{W}^s$,

$$\begin{pmatrix} y_1^s(t) \\ y_2^s(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + \begin{pmatrix} \left(y_1(0) - \frac{1}{3}\right) e^{-3t} \\ 0 \end{pmatrix} \quad t \in [0, \infty),$$

  trace out the saddle path: while $y_2(t) = \bar{y}_2$ stays constant at the steady state level, $y_1^s(t) - \bar{y}_1 = (y_1^s(0) - \bar{y}_1)\, e^{-3t}$ approaches asymptotically its steady state

$$\lim_{t\to\infty} \begin{pmatrix} y_1^s(t) \\ y_2^s(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix},$$

- if $\mathbf{y}(0) \notin \mathcal{E}^-$, that is, if $y_2(0) \neq \frac{1}{3}$, belongs to the unstable manifold, $\mathcal{W}^u$ and

$$\lim_{t\to\infty} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + \left(y_2(0) - \frac{1}{3}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} \frac{1}{3} \\ \pm\infty \end{pmatrix}.$$

Example 3 Now we consider again an homogeneous ODE such that although the coefficient matrix $\mathbf{A}$ is not in a Jordan canonical form but is similar to the coefficient matrices of Examples 1 and 2:

$$\begin{aligned} \dot{y}_1 &= -2y_1 + 5y_2, \\ \dot{y}_2 &= y_1 + 2y_2, \end{aligned} \tag{4.6}$$

where $\mathbf{y} \in \mathrm{Y} = \mathbb{R}^2$. The phase diagram - see Figure 4.3 panel (c) - is built in the same way as in the previous examples. As in Example 1 there is only one steady state in the origin and the steady state is a saddle point. Now the isoclines are $\mathbb{I}_{y_1} = \{\mathbf{y} : -2y_1 + 5y_2 = 0\}$ and $\mathbb{I}_{y_2} = \{\mathbf{y} : y_1 + 2y_2 = 0\}$. Differently from the previous examples, they are not coincident or parallel to the axes, and, as we see next they are not coincident the eigenspaces.

In order to determine analytically the slopes of the eigenspaces and the solutions of the problem we need to make use of our previous results. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} -2 & 5 \\ 1 & 2 \end{pmatrix}.$$

has trace$(\mathbf{A}) = 0$ and det$(\mathbf{A}) = -9$, which yield the same eigenvalues as in Examples 1 and 2: $\lambda_- = -3$ and $\lambda_+ = 3$. Furthermore, the fact det$(\mathbf{A}) \neq 0$ also implies that the steady state, $\mathbf{y} = \mathbf{0}$ is unique.

The eigenvector matrix is now

$$\mathbf{P} = (\mathbf{P}^-, \mathbf{P}^+) = \begin{pmatrix} -5 & 1 \\ 1 & 1 \end{pmatrix}.$$

The (general) solution of the equation, $\mathbf{y}(t) = \mathbf{P}\mathbf{e}^{\boldsymbol{\Lambda}\mathbf{t}}\ \mathbf{w}(0)$, where $\mathbf{w}(0) = \mathbf{P}^{-1}\ \mathbf{y}(0)$, that is

$$\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -y_1(0) + y_2(0) \\ y_1(0) + 5y_2(0) \end{pmatrix}.$$

is

$$\mathbf{y}(t) = w_1(0) \begin{pmatrix} -5 \\ 1 \end{pmatrix} e^{-3t} + w_2(0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t},\ t \in [0, \infty). \tag{4.7}$$

Therefore, the eigenspaces[2] are

$$\mathcal{E}^{-} = \{\mathbf{y} \in \mathrm{Y} : y_1 + 5\, y_2 = 0\} \text{ and } \mathcal{E}^{+} = \{\mathbf{y} \in \mathrm{Y} : y_1 - y_2 = 0\},$$

and the stable manifold is $\mathcal{W}^s = \mathcal{E}^-$.

Again, there are two types of solution paths:

- if $\mathbf{y}(0) \in \mathcal{E}^-$, that is if $w_2(0) = 0$, which is equivalent to having $y_2(0) = \tilde{y_2}(0) = -\frac{1}{5}\, y_1(0)$, the particular solutions are

$$\mathbf{y}^s(t) = y_1(0) \begin{pmatrix} 1 \\ -\frac{1}{5} \end{pmatrix} e^{-3t}\ t \in [0, \infty) \tag{4.8}$$

  because $\tilde{w}_1(0) = -\frac{1}{6}\left(-y_1(0) + \tilde{y_2}(0)\right) = -\frac{1}{5} y_1(0)$. They converge asymptotically to the steady state, $\lim_{t\to\infty} \mathbf{y}^s(t) = \mathbf{0}$. Intuitively, we obtain the saddle path by canceling the explosive effect of e^{3t} on the solution, in order to have only te stabilizing effect of e^{-3t}. Expanding equation (4.8) we have

$$\begin{aligned} y_1^s(t) &= y_1(0)e^{-3t}, \\ y_2^s(t) &= -\frac{1}{5}\ y_1(0)e^{-3t} \end{aligned} \quad \text{for any } \ t \in [0, \infty)$$

  which taking the common element $y_1(0)e^{-3t}$ yields

$$y_1(0)e^{-3t} = y_1^s(t) = -5 y_2^s(t)$$

  which confirms our previous conclusion on the slope of the stable manifold $\mathcal{W}^s$.

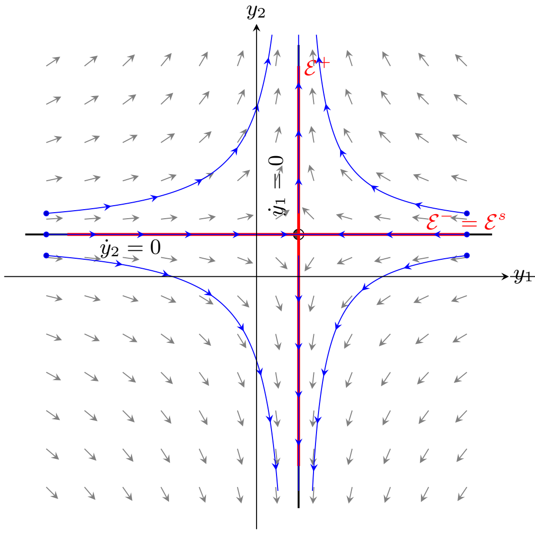
- if $\mathbf{y}(0) \notin \mathcal{E}^-$, that is if $w_2(0) \neq 0$, then

$$\lim_{t\to\infty} \mathbf{y}(t) = w_2(0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} \mp\infty \\ \mp\infty \end{pmatrix},$$

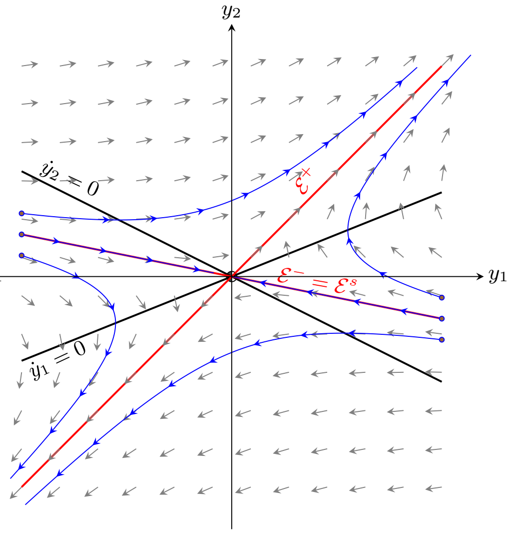
  the solution diverges asymptotically to the direction defined by $\mathcal{E}^+$, that is to a line $y_2(t) = y_1(t)$.

Summing up, going from a matrix the Jordan form to a similar matrix not in the Jordan form introduces a linear transformation on the more important loci, the isoclines and the eigenspaces, consisting in their rotation and changing their relative slopes.

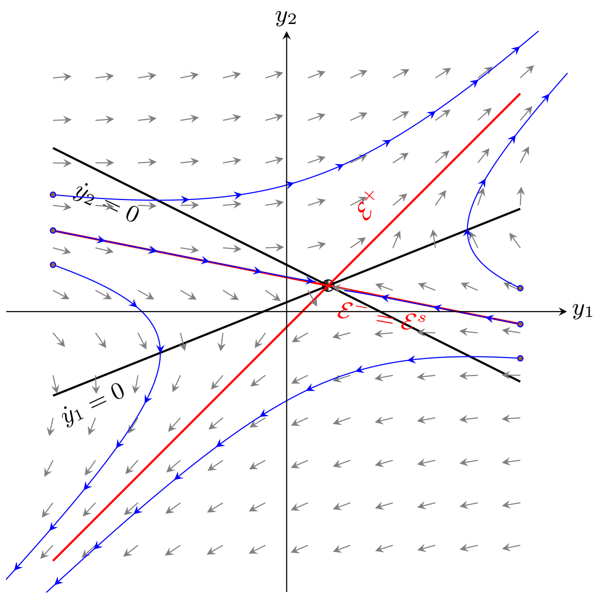
When looking at a phase diagram, one important property of the solutions can be noted: when a trajectory cross one isocline, the related variable change direction. For instance, for trajectories



(a) Phase diagram for example 2



(b) Phase diagram for example 3



(c) Phase diagram for example 4

Figure 4.3: Phase diagrams for Examples 2, 3 and 4

crossing the isocline $\dot{y}_1 = 0$ variable $y_1(t)$ changes from increasing (decreasing) over time to decreasing (increasing) over time. That is, at those points taking derivatives of the solution we will find $\frac{dy_1(t)}{dt} = 0$. The same is valid for $y_2(t)$ when a trajectory crosses isocline $\dot{y}_2 = 0$.

Example 4 Consider the ODE,

$$\begin{aligned}\dot{y}_1 &= -2y_1 + 5y_2 - \frac{1}{5},\\ \dot{y}_2 &= y_1 + 2y_2 - \frac{4}{5}.\end{aligned} \tag{4.9}$$

which is a non-homogeneous case of Example 3. It is a non-homogenous equation of type $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$, where matrix $\mathbf{A}$ is as in example (4.6). The phase diagram is in panel (d) of Figure 4.3. If we compare with the homogeneous case, in panel (c) of the same Figure, we readily observe that there is an upward shift of both isoclines $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$. This is again the same difference we noted when comparing the phase diagrams for Example 1 with Example 2.

Because of that property, we can take the solution of Example 3 and evaluate $\mathbf{y}(t)$ in differences from the new steady state, which is now

$$\bar{\mathbf{y}} = -\mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \end{pmatrix}.$$

Therefore, in this case the general solution is

$$\mathbf{y}(t) = \begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \end{pmatrix} + w_1(0) \begin{pmatrix} -5 \\ 1 \end{pmatrix} e^{-3t} + w_2(0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}, \tag{4.10}$$

where

$$\begin{aligned}\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} &= \mathbf{P}^{-1} \begin{pmatrix} y_1(0) - \bar{y}_1 \\ y_2(0) - \bar{y}_2 \end{pmatrix}\\ &= \frac{1}{6} \begin{pmatrix} -(y_1(0) - \bar{y}_1) + (y_2(0) - \bar{y}_2) \\ (y_1(0) - \bar{y}_1) + 5\,(y_2(0) - \bar{y}_2) \end{pmatrix}\\ &= \frac{1}{6} \begin{pmatrix} -y_1(0) + y_2(0) + \frac{1}{5} \\ y_1(0) + 5\,y_2(0) - \frac{7}{5} \end{pmatrix}.\end{aligned}$$

The eigenspaces are, thus,

$$\mathcal{E}^- = \{\mathbf{y} : y_1 + 5\,y_2 - \frac{7}{5} = 0\},\ \mathcal{E}^+ = \{\mathbf{y} : -y_1 + y_2 + \frac{1}{5} = 0\}$$

The fixed point is again a saddle point and the stable manifold is again $\mathcal{W}^s = \mathcal{E}^-$. The solutions along the saddle path are again obtained by setting $w_2(0) = 0$, yielding

$$\mathbf{y}^s(t) = \begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \end{pmatrix} + \Big(-\frac{1}{5}\ y_1(0) + \frac{2}{15} \Big) \begin{pmatrix} -5 \\ 1 \end{pmatrix} e^{-3t}.$$

The solutions such that $w_2(0) \neq 0$, i.e, in $\mathcal{W}^u$ converge asymptotically to a parametric line $y_2(t) = -\frac{1}{5} + y_1(t)$.

[2]Recall that $\mathcal{E}^- = \{\mathbf{y} \in \mathrm{Y} : -P_2^-(y_1(0) - \bar{y}_1) + P_1^-(y_2(0) - \bar{y}_2) = 0\}$, and $\mathcal{E}^+ = \{\mathbf{y} \in \mathrm{Y} : P_2^+(y_1(0) - \bar{y}_1) - P_1^+(y_2(0) - \bar{y}_2) = 0\}$.

Conclusion

Comparing the phase diagrams for the previous Examples 1, 2, 3 and 4, we conclude:

1. the homogeneous equation for the simplest saddle in which the coefficient matrix **A** is in Jordan canonical form, is a diagonal matrix with non-zero real elements with opposite signs. It displays the crucial elements allowing for a qualitative characterization of the dynamics: there is a stable manifold, $\mathcal{W}^s$, which is a one-dimensional linear manifold [3] in the 2×2 state space Y, with a slope given by the eigenspace associated to the negative eigenvalue, all the flows starting at that manifold will converge to the unique steady state. All the flow starting outside $\mathcal{W}^s$ will become unbounded and their trajectories will be attracted to $\mathcal{E}^+$, which is a one-dimensional linear manifold whose slope is given by the eigenvector associated to the positive eigenvalue. Both manifolds cross at the steady state point, which in this case is the origin;

2. the ODE for a homogeneous saddle, having a coefficient matrix which is not in the canonical form, has the same type of phase diagram. However, it displays a rotation of the isoclines and of both eigenspaces, still crossing at the origin. This is translated geometrically by the fact that the eigenspaces may not be co-incident with the isoclines. Therefore main difference is quantitative, not qualitative;

3. when the ODE is non-homogeneous, i.e., vector $\mathbf{B} \neq \mathbf{0}$, the only significant difference is that the steady state is shifted out of the origin. Now the isoclines and the eigenspaces have the same properties as in the previous two cases but refering to the shifted steady state, not the origin.

From those properties we say that the case in Example 2 is the **normal form** of the saddle, because it is the simples parametric case whose phase diagram is a saddle. However, the same form of matrix **A** is consistent with other phase diagrams if we consider a diagonal matrix with arbitrary real parameters.

Next we consider all the normal forms for planar linear autonomous ODE's.

4.2.3 Nodes without multiplicity and saddles

Differential equations whose geometry is a node or a saddle have the following normal form

$$\dot{y}_1 = \lambda_- \, y_1 + b_1 \tag{4.11a}$$

$$\dot{y}_2 = \lambda_+ \, y_2 + b_2 \tag{4.11b}$$

where we assume that $\lambda_- \ \leq \lambda_+$. The solution, which we know exists and is unique, is a mapping $\mathbf{y} : \mathrm{T} \to \mathrm{Y} \subseteq \mathbf{R}^2$.

[3] A linear manifold corresponds to the set of points (x, y) satisfying the linear equation $a\,x + b\,y = c$ where a, b and c are arbitrary real numbers.

Proposition 1 (Nodes and saddles). *Consider the linear planar ODE specified by equations* (4.11a)-(4.11b) *where* λ_-, λ_+, b_1, *and* b_2 *are all real numbers. Assume that* $\lambda_- \leq \lambda_+$. *Let* $\bar{y}_1 = -\frac{b_1}{\lambda_-}$, *if* $\lambda_- \neq 0$. *and* $\bar{y}_2 = -\frac{b_2}{\lambda_+}$, *if* $\lambda_+ \neq 0$. *A solution exists and is unique and can take the following forms:*

1. *if* $\lambda_- \neq 0$ *and* $\lambda_+ \neq 0$ *the solution is*

$$y_1(t) = \bar{y}_1 + (y_1(0) - \bar{y}_1)\, e^{\lambda_- t}, \tag{4.12a}$$
$$y_2(t) = \bar{y}_2 + (y_2(0) - \bar{y}_2)\, e^{\lambda_+ t}, \tag{4.12b}$$

2. *if* $\lambda_- < 0 = \lambda_+$ *the solution is*

$$y_1(t) = \bar{y}_1 + (y_1(0) - \bar{y}_1)\, e^{\lambda_- t}, \tag{4.13a}$$
$$y_2(t) = y_2(0) + b_2\, t, \tag{4.13b}$$

3. *if* $\lambda_- = 0 < \lambda_+$ *the solution is*

$$y_1(t) = y_1(0) + b_1\, t, \tag{4.14a}$$
$$y_2(t) = \bar{y}_2 + (y_2(0) - \bar{y}_2)\, e^{\lambda_+ t}, \tag{4.14b}$$

4. *if* $\lambda_- = \lambda_+ = 0$ *the solution is*

$$y_1(t) = y_1(0) + b_1\, t, \tag{4.15a}$$
$$y_2(t) = y_2(0) + b_2\, t, \tag{4.15b}$$

where $\mathbf{y}(0) = \left(y_1(0), y_2(0)\right)^\top$ *is an arbitrary element of set* Y.

Proof. As the two differential equations in system (4.12a)-(4.12b) are decoupled, we can apply directly the solutions for the scalar equation. First, consider any j such that $j = 1, 2$ and let $\lambda_j \neq 0$. If we define $z_j(t) = y_j - \bar{y}_j$, where $\bar{y}_j = -\dfrac{b_j}{\lambda_j}$ is the steady state variable y_j, then $\dot{z}_j = \dot{y}_j = \lambda_j\, y_j + b_j = \lambda_j\, (z_j + \bar{y}_j) + b_j = \lambda_j\, z_j$. This scalar ODE has solution $z_j(t) = z_j(0)\, e^{\lambda_j\, t}$. Making the inverse transformation, $y_j(t) = z_j(t) + \bar{y}_j$, we find $y_j(t) = \bar{y}_j + (y_j(0) - \bar{y}_j)\; e^{\lambda_j\, t}$. Second, consider any j such that $j = 1, 2$ and let $\lambda_j = 0$, which yields the differential equation $\dot{y}_j = \dfrac{dy_j(t)}{dt} = b_j$, then $dy_t(t) = b_j\, dt$. Integrating both sides, we find $\int_{y(0)}^{y(t)} dy = \int_0^t b_j\, ds$. Then $y_j(t) - y_j(0) = b_j\, t$. □

Saddles

Let $\lambda_- < 0 < \lambda_+$ in equations (4.11a)-(4.11b). Then there is a unique steady state $\bar{\mathbf{y}} = \left(-\frac{b_1}{\lambda_-}, -\frac{b_2}{\lambda_+}\right)$, at it is a saddle point. It coincides with the origin, $\bar{\mathbf{y}} = \mathbf{0}$, when $\mathbf{B} = \mathbf{0}$. We already presented the phase diagram in Figure 4.2 panel (d), for the case in which $\mathbf{B} = \mathbf{0}$, and in Figure 4.3 panel (a) for the case in which $\mathbf{B} \neq \mathbf{0}$

The (general) solution takes the form equations (4.12a)-(4.12b). The solutions can have two types of asymptotic behavior:

$$\lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{y}} + \lim_{t\to\infty} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (y_1(0) - \overline{y}_1) e^{\lambda_- t} = \overline{\mathbf{y}} \text{ if } \mathbf{y}(0) \neq \overline{\mathbf{y}} \in \mathcal{E}^-$$

or

$$\lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{y}} + \lim_{t\to\infty} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (y_2(0) - \overline{y}_2) e^{\lambda_+ t} = \begin{pmatrix} \overline{y}_1 \\ \pm\infty \end{pmatrix} \text{ if } \mathbf{y}(0) \neq \overline{\mathbf{y}} \in \mathcal{E}^+$$

where $\mathcal{E}^- = \mathcal{W}^s$ is the stable manifold, which in this case is $\mathcal{E}^- = \{\mathbf{y} \in \mathrm{Y} : y_2 = 0\}$. For this reason, we say the solution displays **conditional stability**.

Stable nodes

Let $\lambda_- < \lambda_+ < 0$ in equations (4.11a)-(4.11b). Then there is again is a unique steady state $\bar{\mathbf{y}}$ which is a stable node, coinciding or not with the origin depending on $\mathbf{B}$ being equal to zero or not.

The solution also takes the form of equations (4.12a)-(4.12b). However, for stable nodes the solution has the asymptotic behavior

$$\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{y}(0) \neq \overline{\mathbf{y}} \in \mathrm{Y}.$$

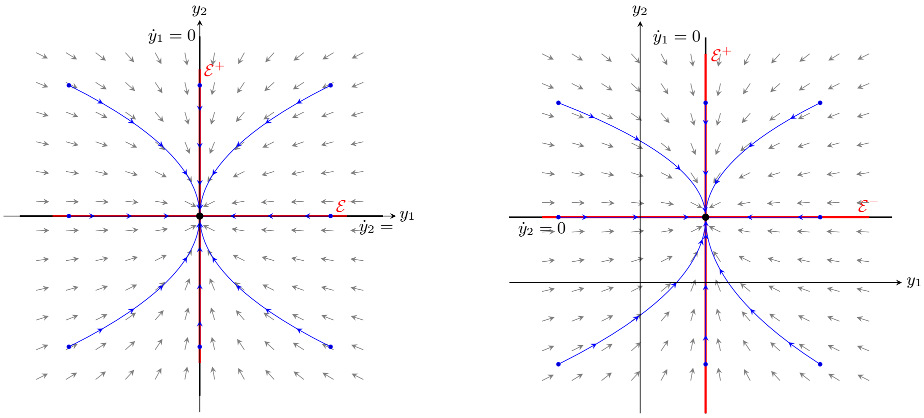
In this case we say that the solution is **asymptotically stable**: all the trajectories converge monotonically to the steady state for any initial point $\mathbf{y}(0) \in \mathrm{Y}$. In this case the whole set Y is an **attractor set** or a stable manifold. It is spanned by the two eigenspaces $\mathcal{E}^-$ and $\mathcal{E}^+$ ($\mathrm{Y} = \mathcal{E}^- \oplus \mathcal{E}^+$).

A representative phase diagrams is in Figure 4.4, which is drawn following the same steps as in Figure 4.2. In this case there are some differences. First, the direction arrows for variable y_2 are directed towards the isocline $\dot{y}_2 = 0$, because the coefficient in that equation is now negative, and not positive as in the case of the saddle. This implies that the vector field points towards the steady state. Second, the slope of the solution in space (y_1, y_2) is

$$\frac{y_2(t)}{y_1(t)} = \frac{y_2(0)}{y_1(0)} e^{(\lambda_+ - \lambda_-)t}, \text{ for } t \in [0, \infty], \tag{4.16}$$

therefore, because $(\lambda_+ - \lambda_-) > 0$ then all the trajectories, $\mathbf{y}(t)$, converge asymptotically to the vertical axis, that is to the eigenspace $\mathcal{E}^+$, which is infinitely sloped. This is natural because, as λ_+ is smaller in absolute value than λ_-, the attracting force of y_1 towards $\bar{y}_1$ is stronger, when starting far away from the steady state, than the attracting force of y_2 towards $\bar{y}_2$. That is

$$\lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{y}} + \lim_{t\to\infty} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (y_2(0) - \overline{y}_2) e^{\lambda_+ t} = \overline{\mathbf{y}}, \text{for any } \mathbf{y}(0) \neq \overline{\mathbf{y}} \in \mathrm{Y}.$$



(a) Stable node for $\mathbf{B} = \mathbf{0}$ (b) Stable node for $\mathbf{B} \neq \mathbf{0}$

Figure 4.4: Phase diagrams for normal form stable nodes

Unstable node

Let $0 < \lambda_- < \lambda_+$ in equations (4.11a)-(4.11b). Again there is a unique steady state $\bar{\mathbf{y}}$ which is an unstable node, which coincides or not with the origin depending on $\mathbf{B}$ being equal to zero or not.

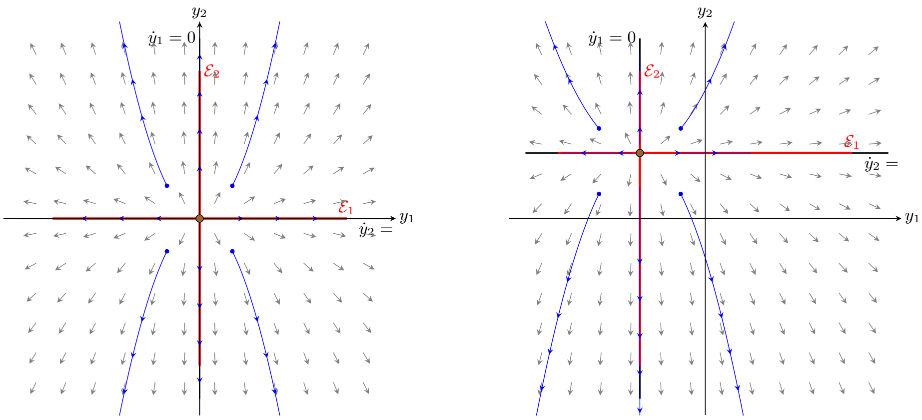
The solution is formally given in equations (4.12a)-(4.12b), and has the asymptotic behavior

$$\lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{y}} + \lim_{t\to\infty} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} (y_1(0) - \overline{y}_1) e^{\lambda_- t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (y_2(0) - \overline{y}_2) e^{\lambda_+ t} \right)$$

$$= \begin{pmatrix} \pm\infty \\ \pm\infty \end{pmatrix}, \text{for any } \mathbf{y}(0) \neq \overline{\mathbf{y}} \in \mathrm{Y}.$$

In this case we say that the solution is **unstable**: any initial deviation from the steady state will generate a flow which is unbounded over time. The phase diagrams is in Figure 4.5. All the trajectories will diverge along the direction of $\mathcal{E}^+$, with maximum strength when they are away from the steady state. In this case all set Y is a **repeller set** because any deviation from the steady state will generate a flow which will be repelled away from it.

The phase diagram 4.5 represents the forward interpretation of the ODE with positive coefficients. However, if we invert the time direction, from forward to backwards, i.e., from $t = 0$ to $t = -\infty$, the solution will be attracted to, or to a neighborhood of, the steady state. This property is sometimes used in economics.



(a) Unstable node for $\mathbf{B} = \mathbf{0}$ (b) Untable node for $\mathbf{B} \neq \mathbf{0}$

Figure 4.5: Phase diagrams for normal form unstable nodes

Stable saddle-nodes

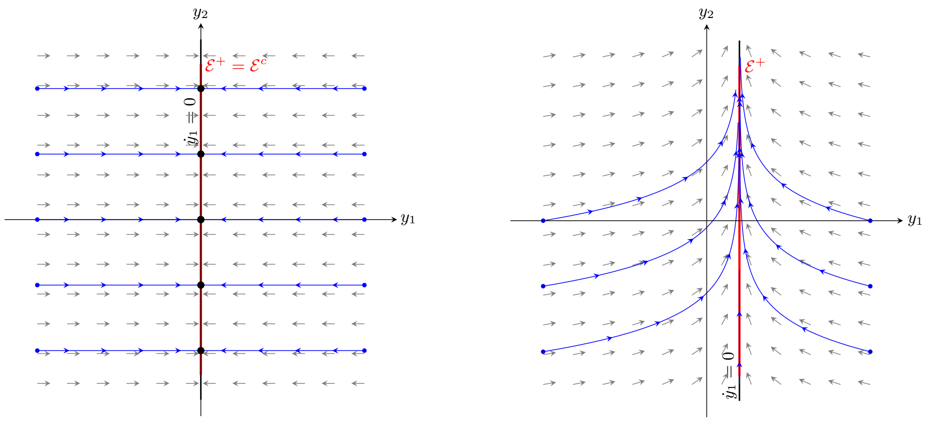
If $\lambda_- < 0 = \lambda_+$ in equations (4.11a)-(4.11b) two cases can occur: first, if $b_2 = 0$ there will be an infinite number of steady states along the line $\bar{y}_1 = -\frac{b_1}{\lambda_-}$; second, if $b_2 \neq 0$ steady states do not exist. In both cases the solution of the ODE is formally given in equations (4.13a)-(4.13b).

In the first case we say there is a stable saddle-node. There is an infinite number of steady states, along the $\dot{y}_1 = 0$ isocline, i.e., for any value of $\mathbf{y}(0)$, the solution converges to a steady state $(\bar{y}_1, y_2)$ where y_2 is arbitrary. Therefore, the eigenspace $\mathcal{E}^+ = \{ \mathbf{y} \in \mathrm{Y} : y_1 = \bar{y}_1\}$ attracts all the trajectories.

Figure 4.6 panel (a) has a representation of the phase diagram for the stable saddle-node. The reason for the name is that this is a boundary case between a saddle, for which $\lambda_+ > 0$, and a stable node, for which $\lambda_+ < 0$. If we compare with Figures 4.4 and 4.5 we observe that that eigenspace $\mathcal{E}^+$ attracts the trajectories that become unbounded asymptotically, for the saddle, and it attracts the trajectories that converge asymptotically to the steady state, for the stable node. Therefore, the case in which $\lambda_+ = 0$ is in the boundary between the saddle and the stable node cases, and we call **center manifold** to the locus of equilibrium points it contains: therefore $\mathcal{W}^c = \{ \mathbf{y} : y_1 = \bar{y}_1\}$.

The change in the parameter close to $\lambda_+ = 0$ is called **unfolding** and we say that $(\lambda_+, \bar{\mathbf{y}}(\lambda_+)) = (0, \bar{\mathbf{y}}(0))$ is a **bifurcation point**.

Figure 4.6 panel (b) shows the phase diagram for the case in which $\lambda_- < 0 = \lambda_+$ and $b_2 \neq 0$. As we can see, in this case a steady state does not exist: in the limit $\lim_{t\to\infty} \mathbf{y}(t) = (\bar{y_1}, \pm\infty)$. All the trajectories, and in particular trajectories in which $y_1(0) = \bar{y}_1$, converge to the eigenspace $\mathcal{E}^+$ and the value of y_2 becomes unbounded: they converge to $+\infty$ if $b_2 > 0$ and to $-\infty$ if $b_2 < 0$.



(a) Stable saddle-node (b) Phase diagram for $\lambda_- < 0 = \lambda_+$ and $b_2 > 0$

Figure 4.6: Phase diagrams case $\lambda_- < 0 = \lambda_+$.

Unstable saddle-nodes

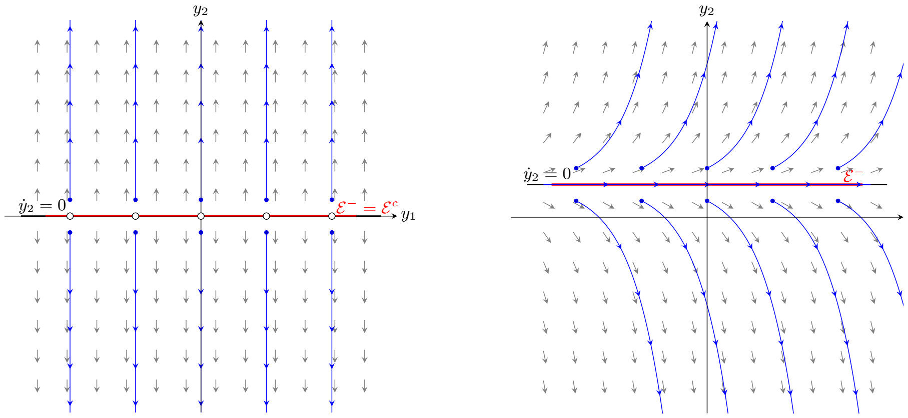
If $\lambda_- = 0 < \lambda_+$ in equations (4.11a)-(4.11b) two cases can occur: first, if $b_1 = 0$ there will be an infinite number of steady states along the line $\overline{y}_2 = -\frac{b_2}{\lambda_+}$; second, if $b_1 \neq 0$ steady states do not exist. In both cases the solution of the ODE is formally given in equations (4.14a)-(4.14b).

In the first case we say there is an unstable saddle-node. There is an infinite number of steady states, along the $\dot{y}_2 = 0$ isocline. If the initial point satisfies $\mathbf{y}(0) = (y_1(0), \bar{y}_2)$ the solution is stationary, that is it remains constant. However, any deviation of $y_2(0)$ from $\bar{y}_2$ will generate a trajectory that becomes asymptotically unbounded. Therefore there will be an infinite number of unstable steady states along the line $\dot{y}_2 = 0$, which co-incides with the eigenspace $\mathcal{E}^- = \{\mathbf{y} \in \mathrm{Y} : y_2 = \bar{y}_2\}$.

Figure 4.7 panel (a) has a representation of the phase diagram for the unstable saddle-node. The reason for the name is that this is a boundary case between a saddle, for which $\lambda_- < 0$, and an unstable node, for which $\lambda_- > 0$. If we compare with Figures 4.4 and 4.5 we observe that that eigenspace $\mathcal{E}^-$ defines a direction that repels the trajectories that become unbounded asymptotically, for the saddle, and it also defines a direction that repels the trajectories that diverge asymptotically from the steady state, for the unstable node. Therefore, the case in which $\lambda_- = 0$ separates the saddle from the unstable node cases, and we call again **center manifold** to the locus of equilibrium points if contains: therefore $\mathcal{W}^c = \{\ \mathbf{y} : y_2 = \bar{y}_2\}$.

The change in the parameter close to $\lambda_- = 0$ is called **unfolding** and we say that $(\lambda_-, \overline{\mathbf{y}}(\lambda_-)) = (0, \overline{\mathbf{y}}(0))$ is a bifurcation point.

Figure 4.7 panel (b) shows the phase diagram for the case in which $\lambda_- = 0 < \lambda_+$ and $b_1 \neq 0$. As we can see, in this case a steady state does not exist: the trajectories tend to diverge away from $\mathcal{E}^-$.



(a) Unstable saddle-node

(b) Phase diagram for $\lambda_- = 0 < \lambda_+$ and $b_1 > 0$

Figure 4.7: Phase diagrams case $\lambda_- = 0 < \lambda_+$.

Degenerate saddle-nodes

A degenerate saddle-node exists if $\lambda_- = \lambda_+ = 0$. The solution also takes the form of equations (4.15a)-(4.15b). It is easy to see that three cases can occur:

1. if $b_1 = b_2 = 0$ then the solution is degenerate along the two dimensions,

$$\mathbf{y}(t) = \mathbf{y}(0), \text{ for any } t \in [0, \infty)$$

   that is, the solution is stationary for any arbitrary value $\mathbf{y}(0) \in \mathrm{Y}$. There is essentially no dynamics. In the case all the state space, Y, can be seen as a center manifold: $\mathcal{W}^c = \mathrm{Y}$. This is the highest level of degeneracy that we can have. Furthermore, this case can be seen as a degenerate case in the boundary of all possible phase diagrams for a planar linear ODE;

2. if $b_1 = 0$ and $b_2 \neq 0$, or $b_1 \neq 0$ and $b_2 = 0$ a steady state does not exist. However, while one of the variables (y_1 in the first case and y_2 in the second case) will be constant over time, the other will become asymptotically unbounded (y_2 in the first case and y_1 in the second case) and follows a linear progression over time;

3. if $b_1 \neq 0$ and $b_2 \neq 0$ steady states does not exist as well. However, in this case both variables diverge asymptotically.

4.2.4 Nodes with multiplicity

Nodes with multiplicity are the geometric representation of planar linear ODE in which matrix $\mathbf{A}$ has discriminant equal to zero, that is when $\mathbf{A}$ has the Jordan canonical form $\mathbf{\Lambda}_2$. In this case the normal form of the ODE is the following:

$$\dot{y}_1 = \lambda\, y_1 + y_2 + b_1 \tag{4.17a}$$

$$\dot{y}_2 = \lambda\, y_2 + b_2. \tag{4.17b}$$

The general solution of this ODE is provided by the following proposition:

Proposition 2. *Consider the linear planar ODE defined by equations* (4.17a)-(4.17b) *where* λ, b_1 *and* b_2 *are real numbers. A solution exists and is unique and can take the following forms:*

1. *If* $\lambda \neq 0$ *the solution is*

$$y_1(t) = \bar{y}_1 + \Big(y_1(0) - \bar{y}_1 + (y_2(0) - \bar{y}_2)\, t\Big)\, e^{\lambda\, t} \tag{4.18a}$$

$$y_2(t) = \bar{y}_2 + (y_2(0) - \bar{y}_2)\, e^{\lambda\, t} \tag{4.18b}$$

where the steady state is

$$\bar{\mathbf{y}} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} -\Big(\frac{b_1}{\lambda} - \frac{b_2}{\lambda^2}\Big) \\ -\frac{b_2}{\lambda} \end{pmatrix} \tag{4.19}$$

2. *if If* $\lambda = 0$ *the solution is*

$$y_1(t) = y_1(0) + (y_2(0) + b_1)\, t + \frac{b_2}{2}\, t^2 \tag{4.20a}$$

$$y_2(t) = y_2(0) + b_2 t \tag{4.20b}$$

Proof. First consider the case in which $\lambda \neq 0$. Using the same method as in the proof of Proposition 1 we find the solution of the ODE (4.17b) to be

$$y_2(t) = -\frac{b_2}{\lambda} + (y_2(0) + \frac{b_2}{\lambda})\, e^{\lambda\, t}.$$

Substituting in equation (4.17a) yields the scalar linear non-autonomous ODE

$$\dot{y}_1 = \lambda\, y_1 + b_1 - \frac{b_2}{\lambda} + \Big(y_2(0) + \frac{b_2}{\lambda}\Big)\, e^{\lambda\, t}.$$

Integrating, we find

$$\begin{aligned}
y_1(t) &= e^{\lambda\, t}\left(y_1(0) + \int_0^t e^{-\lambda\, s}\left(b_1 - \frac{b_2}{\lambda} + \Big(y_2(0) + \frac{b_2}{\lambda}\Big)\, e^{\lambda\, s}\right) ds\right) \\
&= e^{\lambda\, t}\left(y_1(0) + \int_0^t e^{-\lambda\, s}\Big(b_1 - \frac{b_2}{\lambda}\Big)\, ds + \int_0^t \Big(y_2(0) + \frac{b_2}{\lambda}\Big) ds\right) \\
&= e^{\lambda\, t}\left(y_1(0) + \Big(b_1 - \frac{b_2}{\lambda}\Big)\frac{1}{\lambda}\Big(e^{-\lambda\, t} - 1\Big) + \Big(y_2(0) + \frac{b_2}{\lambda}\Big)\, t\right) \\
&= -\Big(b_1 - \frac{b_2}{\lambda}\Big)\frac{1}{\lambda} + \left(y_1(0) + \Big(b_1 - \frac{b_2}{\lambda}\Big)\frac{1}{\lambda} + \Big(y_2(0) + \frac{b_2}{\lambda}\Big)\, t\right) e^{\lambda\, t}.
\end{aligned}$$

As at a steady state $\dot{\mathbf{y}} = \mathbf{0}$, writing the system (4.17a)-(4.17b) in matrix notation, we find

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \bar{\mathbf{y}} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Solving for $\mathbf{y}$ we find the steady state as in equation (4.19), which means that the solution can be written as in equations (4.18a)-(4.18b).

Now, let $\lambda = 0$. The solution of the ODE (4.17b) is $y_2(t) = y_2(0) + b_2\, t$ which implies that equation (4.17a) becomes $\dot{y}_1 = b_1 + y_2(0) + b_2\, t$, which has solution $y_1(t) = y_1(0) + (b_1 + y_2(0))\, t + \frac{b_2}{2}\, t^2$. □

Stable node with multiplicity

If $\lambda < 0$, in the planar linear ODE (4.17a)-(4.17b), then there is an unique steady state $\overline{\mathbf{y}} = \left(-\left(\frac{b_1}{\lambda} - \frac{b_2}{\lambda^2} \right), -\frac{b_2}{\lambda} \right)$, independently of the vector $\mathbf{B}$. The solutions are given in equations (4.18a)-(4.18b).

The geometric representation of the dynamics is a stable node with multiplicity, which is depicted in 4.8 panel (a) for the case in which $\mathbf{B} = \mathbf{0}$. We see that all trajectories converge to a direction defined by the simple eigenvalue $\mathcal{E}^s = \{ \mathbf{y} \in \mathrm{Y} : y_2 = \overline{y}_2 \}$ (see the Appendix to chapter 3) in their convergence towards the steady state.

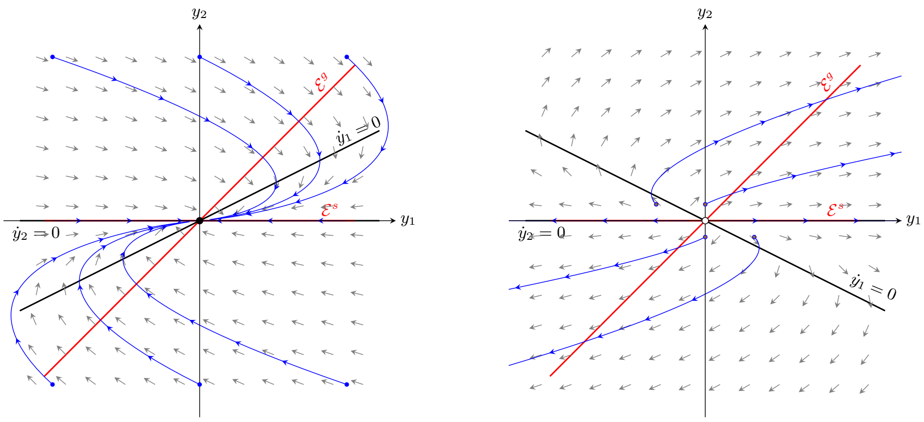
Differently from the stable node, instead of the existence of convergence to four potential directions of approximation to the steady state, in this case there are only two directions of approximation, one for trajectories starting from positive initial values for y_2 and another for trajectories starting from negative initial values of y_2. This is the main consequence of the multiplicity of the steady states.

This implies most trajectories are **hump-shaped**: while $y_2(t)$ converges monotonically to $y_2(\infty) = \overline{y}_2$, variable y_1, particularly if the initial point starts from a point in which $y_2(0)$ is very different from $\overline{y}_2$, tends to change direction in the transition to the steady state (see 4.8 panel (a)), when it crosses the $\dot{y}_1 = 0$ isocline.

Unstable node with multiplicity

If $\lambda > 0$, in the planar linear ODE (4.17a)-(4.17b), then there is an unique steady state $\overline{\mathbf{y}} = \left(-\left(\frac{b_1}{\lambda} - \frac{b_2}{\lambda^2} \right), -\frac{b_2}{\lambda} \right)$, independently of the vector $\mathbf{B}$. The solutions are formally given in equations (4.18a)-(4.18b).

The geometric representation of the dynamics is a stable node with multiplicity, which is depicted in 4.8 panel (b) for the case in which $\mathbf{B} = \mathbf{0}$. We see, again, that all trajectories converge to a direction defined by the simple eigenvalue $\mathcal{W}^s = \{ \mathbf{y} \in \mathrm{Y} : y_2 = \overline{y}_2 \}$, in their increasing deviation from the steady state. As the the stable case some unstable trajectories can be hump-shaped when they cross the $\dot{y}_2 = 0$ isocline.



(a) Stable node with multiplicity (b) Unstable node with multiplicity

Figure 4.8: Phase diagram nodes with multiplicity and $\mathbf{B} = \mathbf{0}$

Degenerate node with multiplicity

If $\lambda = 0$, in the planar linear ODE (4.17a)-(4.17b), the formal solutions are given in equations (4.20a)-(4.20b), there are three possible cases as regards the dynamics of the solution:

1. if $b_1 = b_2 = 0$ then the solution is $\mathbf{y}(t) = \mathbf{y}(0)$ for any $t \in [0, \infty)$, that is, it is stationary. This means that there is an infinite number of steady states, as in the degenerate saddle-node;
2. if $b_1 \neq 0$ and $b_2 = 0$ a steady state does not exist, although y_2 is stationary, because $y_2(t) = y_2(0)$. The other variable changes over time in a linear way, for any $y_1(0)$, because $y_1(t) = y_1(0) + b_1 t$;
3. if $b_2 \neq 0$, for any b_1 there is no steady state and the solution will change over time for both variables.

4.2.5 Foci

Foci are the geometric representation of planar linear ODE in which matrix $\mathbf{A}$ has a negative valued discriminant, that is when $\mathbf{A}$ has the Jordan canonical form $\mathbf{\Lambda}_3$. In this case the normal form of the ODE is the following:

$$\dot{y}_1 = \alpha\, y_1 + \beta\, y_2 + b_1 \tag{4.21a}$$
$$\dot{y}_2 = -\beta\, y_1 + \alpha\, y_2 + b_2 \tag{4.21b}$$

Proposition 3 (Foci). *Consider the linear planar ODE defined by equations* (4.21a)-(4.21b) *where* α*, and* $\beta \neq 0$ *are real numbers. A solution exists and is unique and can take the following forms:*

1. *if* $\alpha \neq 0$ *the solution is*

$$y_1(t) = \bar{y}_1 + e^{\alpha\, t}\, \Big((y_1(0) - \bar{y}_1)\, \cos(\beta\, t) + (y_2(0) - \bar{y}_2)\, \sin(\beta\, t) \Big) \tag{4.22a}$$

$$y_2(t) = \bar{y}_2 + e^{\alpha\, t}\, \Big(-\, (y_1(0) - \bar{y}_1)\, \sin(\beta\, t) + (y_2(0) - \bar{y}_2)\, \cos(\beta\, t) \Big) \tag{4.22b}$$

where the steady state is

$$\bar{\mathbf{y}} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = -\frac{1}{\alpha^2 + \beta^2} \begin{pmatrix} \alpha\, b_1 - \beta\, b_2 \\ \beta\, b_1 + \alpha\, b_2 \end{pmatrix} \tag{4.23}$$

2. *if* $\alpha = 0$ *the solution is*

$$y_1(t) = \bar{y}_1 + (y_1(0) - \bar{y}_1)\, \cos(\beta\, t) + (y_2(0) - \bar{y}_2)\, \sin(\beta\, t) \tag{4.24a}$$

$$y_2(t) = \bar{y}_2 - (y_1(0) - \bar{y}_1)\, \sin(\beta\, t) + (y_2(0) - \bar{y}_2)\, \cos(\beta\, t) \tag{4.24b}$$

where the steady state is

$$\bar{\mathbf{y}} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} \dfrac{b_2}{\beta} \\ -\dfrac{b_1}{\beta} \end{pmatrix}. \tag{4.25}$$

Proof. In this case we cannot solve each equation independently, as for the decoupled system in Proposition 1 , or the recursive system in Proposition 2.

First, we transform the non-homogenous system (4.21a)-(4.21b) into a homogeneous system by defining

$$\begin{aligned} z_1(t) &= y_1(t) - \bar{y}_1, \\ z_2(t) &= y_2(t) - \bar{y}_2, \end{aligned} \tag{4.26}$$

where the steady state, $\bar{\mathbf{y}}$ in equation (4.23), is obtained by solving

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Taking time derivatives of (4.26) we have the homogenous ODE

$$\begin{aligned} \dot{z}_1 &= \alpha\, z_1 + \beta\, z_2 \\ \dot{z}_2 &= -\beta\, z_1 + \alpha\, z_2. \end{aligned}$$

Next, we transform this planar ODE into an equivalent system of decoupled ODE (see the Appendix to this chapter). We do this by passing from cartesian coordinates $(z_1, z_2) \in \mathbb{R}$ to polar coordinates $(r, \theta) \in \mathbb{R}$, through the transformation:

$$\begin{aligned} z_1(t) &= r(t) \cos(\theta(t)), \\ z_2(t) &= r(t) \sin(\theta(t)), \end{aligned} \tag{4.27}$$

where $r^2 = z_1^2 + z_2^2$ measures the distance from a reference point (the radius) and θ, is the angular coordinate such that $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{z_1}{z_2}$, that is $\theta = \arctan\left(\dfrac{z_1}{z_2}\right)$.

$$\begin{pmatrix} r(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} \sqrt{z_1(t)^2 + z_2(t)^2} \\ \arctan\left(\dfrac{z_1}{z_2}\right). \end{pmatrix}$$

Taking time-derivatives for both equations, yields

$$\dot{r} = \frac{1}{2}(z_1^2 + z_2^2)^{\frac{1}{2}-1}\left(2\,z_1\,\dot{z}_1 + 2\,z_2\,\dot{z}_2\right) = \frac{z_1\,\dot{z}_1 + z_2\,\dot{z}_2}{r} = \alpha\frac{z_1^2 + z_2^2}{r} = \alpha\,\frac{r^2}{r} = \alpha\,r,$$

and [4]

$$\dot{\theta} = \frac{z_2\,\dot{z}_1 - z_1\,\dot{z}_2}{z_1^2 + z_2^2} = -\beta\,\frac{z_1^2 + z_2^2}{z_1^2 + z_2^2} = -\beta.$$

Solving the two linear decoupled differential equations $\dot{r} = -\alpha r$ and $\dot{\theta} = -\beta$, we find

$$\begin{aligned} r(t) &= r(0)\,e^{\alpha\,t} \\ \theta(t) &= \theta(0) - \beta\,t. \end{aligned}$$

Using equation (4.27) for the inverse transformation, and observing that $z_1(0) = r(0)\,\cos(\theta(0))$ and $z_2(0) = r(0)\,\sin(\theta(0))$, we find [5]

$$\begin{aligned} z_1(t) &= e^{\alpha\,t}\,r(0)\,\cos(\theta(0) - \beta\,t) \\ &= e^{\alpha\,t}\Big(r(0)\,\cos(\theta(0))\,\cos(\beta\,t) \quad + r(0)\,\sin(\theta(0))\,\sin(\beta\,t)\Big) \\ &= e^{\alpha\,t}\Big(z_1(0)\,\cos(\beta\,t) \quad + z_2(0)\,\sin(\beta\,t)\Big) \end{aligned}$$

and

$$\begin{aligned} z_2(t) &= e^{\alpha\,t}\,r(0)\,\sin(\theta(0) - \beta\,t) \\ &= e^{\alpha\,t}\Big(- r(0)\,\cos(\theta(0))\,\sin(\beta\,t) \quad + r(0)\,\sin(\theta(0))\,\cos(\beta\,t)\Big) \\ &= e^{\alpha\,t}\Big(- z_1(0)\,\sin(\beta\,t) \quad + z_2(0)\,\cos(\beta\,t)\Big). \end{aligned}$$

If we apply the inverse transformation of (4.26) we obtain the solution to the differential equation (4.24a)-(4.24b). □

Stable focus

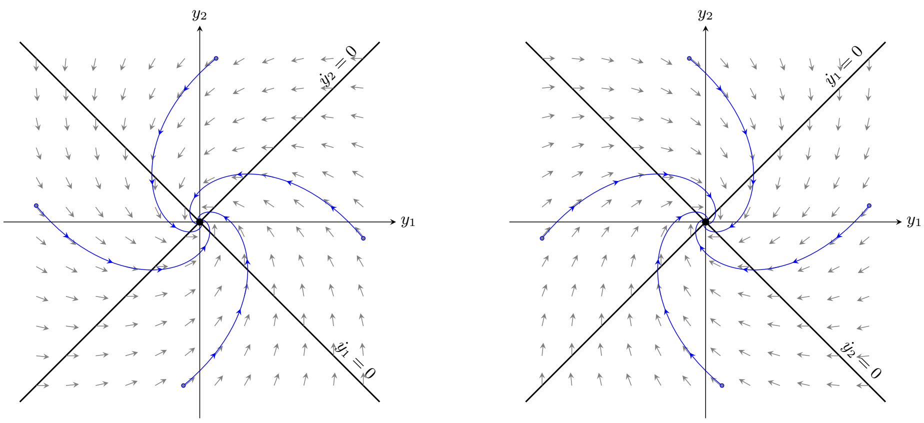
If $\alpha < 0$ and $\beta \neq 0$, in the planar linear ODE (4.21a)-(4.21b), then there is a unique steady state, given in equation (4.23), for any vector $\mathbf{B}$. The solutions are formally given in equations (4.22a)-(4.22b). We can see, because $\lim_{t\to\infty}\ e^{\alpha t} = 0$, that, for any initial value $\mathbf{y}(0)$, the solution converges

[4]The derivative of $\arctan\left(\dfrac{f(x)}{g(x)}\right)$ is $\frac{d}{dx}\left(\arctan\left(\dfrac{f(x)}{g(x)}\right)\right) = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{f(x)^2 + g(x)^2}$.

[5]Recall the following trigonometric equivalences: $\cos(x - y) = \cos(x)\,\cos(y) + \sin(x)\,\sin(y)$ and $\sin(x - y) = -\cos(x)\,\sin(y) + \sin(x)\,\cos(y)$.

asymptotically to the steady state. This is also a case in which there is (global) asymptotic stability, but differently from the stable node, the trajectories are oscillatory (or at least hump shape). Figure 4.9 displays two phase diagrams for the stable focus: panel (a) shows the anti-clockwise case in which $\beta < 0$ and panel (b) shows the clockwise case in which $\beta > 0$. In both cases trajectories are oscillatory, but they can be hump-shaped if the initial point is close to the steady state and a complete periodic trajectory is only materialized for one of the variables.

For the stable focus the steady state is, therefore, an attractor, meaning that the stable manifold is coincident with the domain Y, $\mathcal{W}^s = \mathrm{Y}$, and both the center and the unstable manifolds are empty.



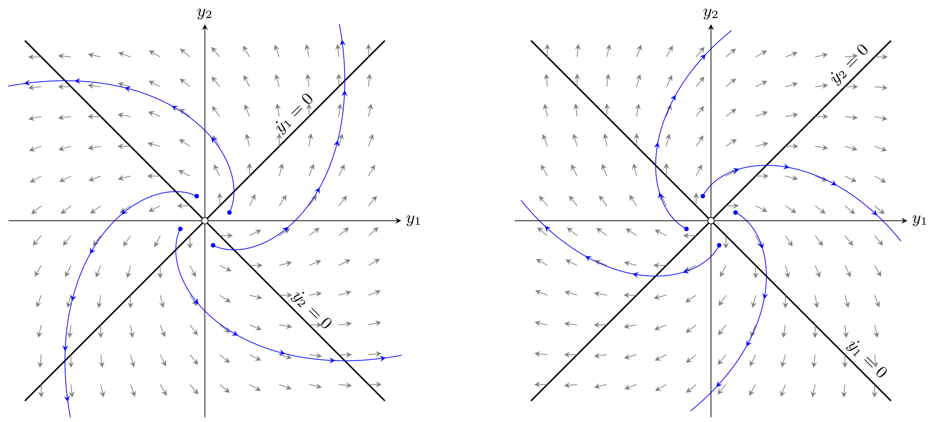
(a) Case $\beta < 0$ (b) Case $\beta > 0$

Figure 4.9: Phase diagrams for stable foci with $\mathbf{B} = \mathbf{0}$.

Unstable focus

If $\alpha > 0$ and $\beta \neq 0$, in the planar linear ODE (4.21a)-(4.21b), then there is a unique steady state, given in equation (4.23), for any vector $\mathbf{B}$. The solutions are formally given in equations (4.22a)-(4.22b). We can see, because $\lim_{t\to\infty} e^{\alpha t} = \infty$, that, for any initial value $\mathbf{y}(0)$ different from the steady state, the solution becomes unbounded in infinite time. This is also a case in which there is (global) instability, but differently from the unstable node, the trajectories are oscillatory (or at least hump shape). Figure 4.10 displays two phase diagrams for the unstable focus: panel (a) shows the anti-clockwise case in which $\beta < 0$ and panel (b) shows the clockwise case in which $\beta > 0$. In both cases trajectories are oscillatory, but they can be hump-shaped if the initial point is close to the steady state and a complete periodic trajectory is only materialized for one of the variables. In

For the stable focus the steady state is a repeller, meaning that the unstable manifold is coincident with the domain Y, $\mathcal{W}^u = \mathrm{Y}$, and both the center and the stable manifolds are empty.



(a) Case $\beta < 0$ (b) Case $\beta > 0$

Figure 4.10: Phase diagrams for unstable foci with $\mathbf{B} = \mathbf{0}$.

Center

If $\alpha = 0$ and $\beta \neq 0$, in the planar linear ODE (4.21a)-(4.21b), then there is an unique steady state presented in equation (4.25), independently of the vector $\mathbf{B}$. The solutions are formally given in equations (4.24a) -(4.24b). If $\mathbf{y}(0) \neq \overline{\mathbf{y}}$, we can see that the solution is **periodic**, meaning that $\mathbf{y}(t) = \mathbf{y}(t+p)$, for any $t \in \mathrm{T}$, where p is the amount of time required for a repetition of the solution. This means that the solution is stable but not asymptotically stable: the distance between $\mathbf{y}(0)$ and $\overline{\mathbf{y}}$ is constant, that is it neither converges to zero nor becomes unbounded in infinite time.

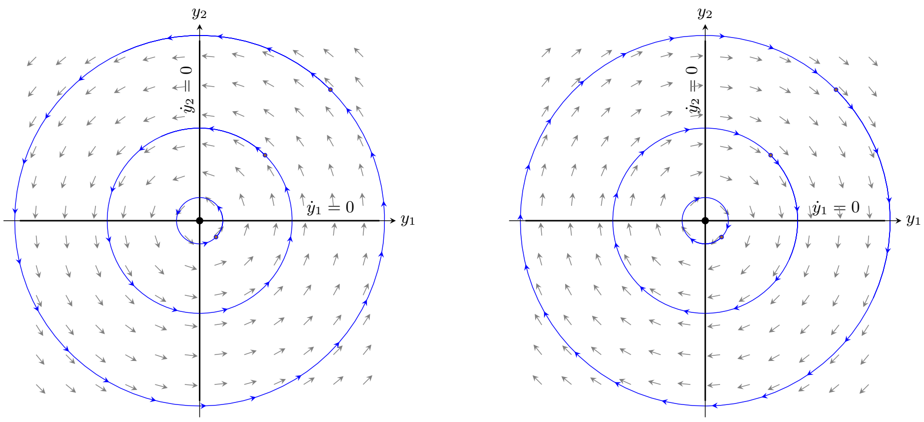
Figure 4.11 displays two phase diagrams for the center: panel (a) shows the anti-clockwise case in which $\beta < 0$ and panel (b) shows the clockwise case in which $\beta > 0$.

In this case the stable and unstable manifolds are both empty and the state space coincides with the center manifold, $\mathcal{W}^c = \mathrm{Y}$.

4.2.6 Non-canonical cases

In this subsection we present the solutions and the phase diagrams when matrix $\mathbf{A}$ is non-canonical. Differently from the previous cases, this are not normal form cases, in the sense that they represent the simplest cases for similar matrices, that is they represent irreducible cases.

Case Λ_d



(a) Case $\beta < 0$ (b) Case $\beta > 0$

Figure 4.11: Phase diagrams for centers, if $\mathbf{B} = \mathbf{0}$.

If the coefficient matrix is the non-canonical case $\mathbf{\Lambda}_d$ the ODE system i

$$\dot{y}_1 = \lambda\, y_1 + b_1, \tag{4.28a}$$

$$\dot{y}_2 = \lambda\, y_2 + b_2. \tag{4.28b}$$

This appears to be similar to a node, in the sense that the two equations are uncoupled, but it is not because the two coefficients affecting the variables y_1 and y_2 are equal. But they differ from the node with multiplicity because the coefficient matrix is diagonal.

The solution is similar to the one of a node

$$y_1(t) = \bar{y}_1 + (y_1(0) - \bar{y}_1)\, e^{\lambda\, t}, \tag{4.29a}$$

$$y_2(t) = \bar{y}_2 + (y_2(0) - \bar{y}_2)\, e^{\lambda\, t}, \tag{4.29b}$$

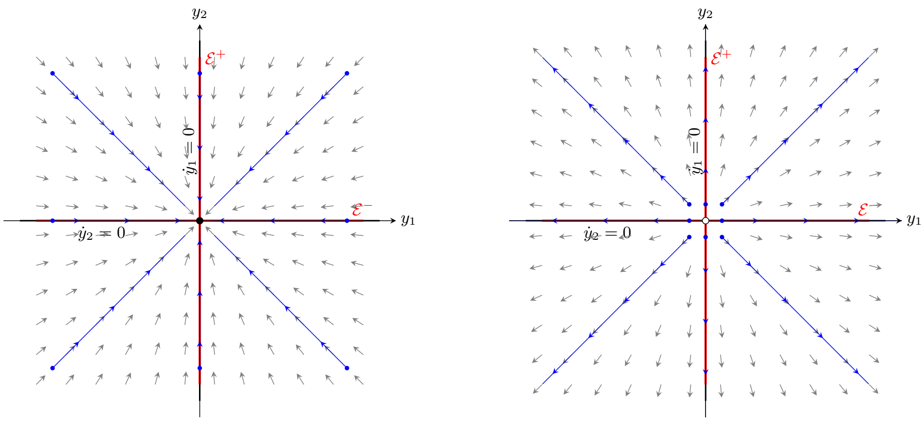
for $\lambda \neq 0$ [6] where

$$\overline{y}_j \ = -\frac{b_j}{\lambda}, \text{ for } \ j = 1, 2.$$

If $\lambda \neq 0$ the steady state alway exists and is unique and the solutions, in equations (4.29a)-(4.29b) are similar to the solutions for (non-degenerate) nodes. If $\lambda = 0$ this case is the same as a degenerate node.

If $\lambda < 0$ the solutions are asymptotically stable and if $\lambda > 0$ they are unstable. Comparing the phase diagram for the stable (unstable) case in Figure 4.12 with the phase diagram for the stable (unstable) node in Figure 4.4 (4.5) the difference is obvious: the trajectories tend to be coincident or equidistant with the two eigenspaces for all times. The qualitative dynamic properties tend to be the same as for the stable or unstable nodes.

[6] If $\lambda = 0$ this reduces to the case of a degenerate saddle-node.



(a) Case $\lambda < 0$ (b) Case $\lambda > 0$

Figure 4.12: Phase diagrams for a non-canonical d-ODE for $\mathbf{B} = \mathbf{0}$.

Case Λ_h: hyperbolic case

The ODE for the hyperbolic case is

$$\dot{y}_1 = \alpha\, y_1 + \beta\, y_2 + b_1 \tag{4.30a}$$

$$\dot{y}_2 = \beta\, y_1 + \alpha\, y_2 + b_2, \tag{4.30b}$$

where $\beta \neq 0$.

Proposition 4 (Non-canonical case $\mathbf{\Lambda}_h$). *Consider the linear planar ODE defined by equations* (4.30a)-(4.30b) *where α and $\beta \neq 0$ are real numbers. A solution exists and is unique and can take the following forms:*

1. *if $\alpha \neq 0$ the solution is* [7]

$$y_1(t) = \bar{y}_1 + e^{\alpha\, t}\, \Big(\big(y_1(0) - \bar{y}_1\big)\, \cosh(\beta\, t) + \big(y_2(0) - \bar{y}_2\big)\, \sinh(\beta\, t) \Big) \tag{4.31a}$$

$$y_2(t) = \bar{y}_2 + e^{\alpha\, t}\, \Big(\big(y_1(0) - \bar{y}_1\big)\, \sinh(\beta\, t) + \big(y_2(0) - \bar{y}_2\big)\, \cosh(\beta\, t) \Big) \tag{4.31b}$$

where the steady state is

$$\bar{\mathbf{y}} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \frac{1}{\alpha^2 - \beta^2} \begin{pmatrix} -\alpha\, b_1 + \beta\, b_2 \\ \beta\, b_1 - \alpha\, b_2 \end{pmatrix}. \tag{4.32}$$

[7] Recall that $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$.

Proof. We follow, again, three steps. First we find the steady state $\bar{\mathbf{y}}$, by solving

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and obtain (4.32). Second, we define the deviations $z_1(t) = y_1(t) - \bar{y}_1$, $z_2(t) = y_2(t) - \bar{y}_2$, and take the time-derivatives to find the variational ODE

$$\begin{aligned} \dot{z}_1 &= \alpha\, z_1 + \beta\, z_2 \\ \dot{z}_2 &= \beta\, z_1 + \alpha\, z_2 \end{aligned}$$

which is again a system of coupled variables. Third, we transform this system into a system of decoupled variables by finding a suitable linear transformation. In this case, the transformation is

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} z_1(t) - z_2(t) \\ z_1(t) + z_2(t) \end{pmatrix}.$$

Taking time derivatives we obtain the decoupled system

$$\begin{aligned} \dot{w}_1 &= (\alpha - \beta)\, w_1 \\ \dot{w}_2 &= (\alpha + \beta)\, z_2 \end{aligned}$$

which has a unique solution

$$\begin{aligned} w_1(t) &= w_1(0)\, e^{(\alpha-\beta)\, t} = \big(z_1(0) - z_2(0)\big)\, e^{\alpha\, t}\ e^{-\beta\, t} \\ w_2(t) &= w_2(0)\, e^{(\alpha+\beta)\, t} = \big(z_1(0) + z_2(0)\big)\, e^{\alpha\, t}\ e^{\beta\, t}. \end{aligned}$$

Using the inverse transformation

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} w_1(t) + w_2(t) \\ -w_1(t) + w_2(t) \end{pmatrix}$$

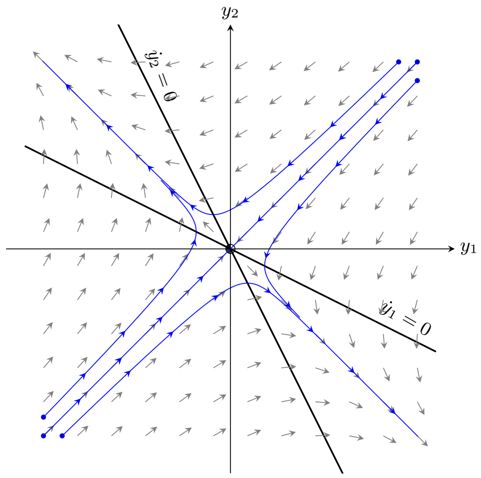
we find

$$\begin{aligned} z_1(t) &= e^{\alpha\, t}\, \frac{1}{2}\ \Big(z_1(0)\, (e^{\beta\, t} + e^{-\beta\, t}) + z_2(0)\, (e^{\beta\, t} - e^{-\beta\, t})\Big) \\ z_2(t) &= e^{\alpha\, t}\, \frac{1}{2}\Big(z_1(0)\, (e^{\beta\, t} - e^{-\beta\, t}) + z_2(0)\, (e^{\beta\, t} + e^{-\beta\, t})\Big). \end{aligned}$$

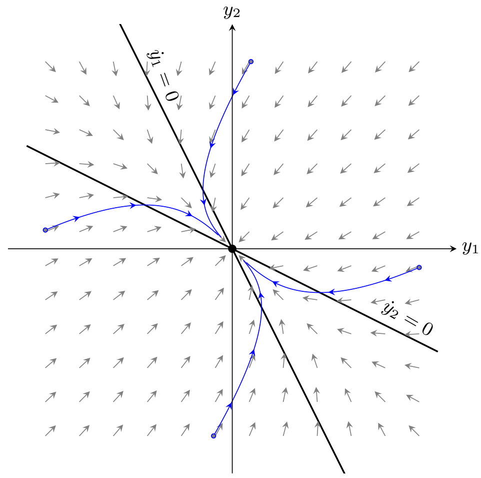
Transforming back to $\mathbf{y}$ and using the definitions of $\cosh(x)$ and $\sinh(x)$ we find the solution (4.31a)-(4.31b). □

The dynamics is determined by the fact that the coefficient matrix $\mathbf{\Lambda}_h$ has eigenvalues $\lambda_- = \alpha - \beta$ and $\lambda_+ = \alpha + \beta$. Therefore they depend on both the absolute value of the coefficients and on their sign.

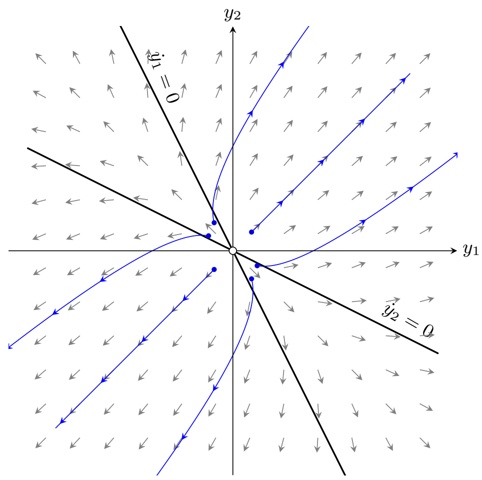
Let us start by assuming that $\mathbf{B} = \mathbf{0}$. In this case a steady state always exists although it may not be unique. The following cases are possible:



(a) Saddle case: $|\beta| > 0$ and $-|\beta| < |\alpha| < |\beta|$



(b) Stable case: $\alpha < 0$ and $-\alpha < \beta < \alpha$



(c) Unstable case: $\alpha > 0$ and $-\alpha < \beta < \alpha$

Figure 4.13: Phase diagrams for a non-canonical h-ODE for $\mathbf{B} = \mathbf{0}$.

1. if $|\beta| > 0$ and $-|\beta| < |\alpha| < |\beta|$ then the steady state $\bar{\mathbf{y}} = \mathbf{0}$ is unique and is a saddle-point (see panel (a) in Figure 4.13);

2. if $\alpha < 0$ and $-\alpha < \beta < \alpha$ then the steady state $\bar{\mathbf{y}} = \mathbf{0}$ is unique is asymptotically stable (see panel (b) in Figure 4.13);

3. if $\alpha > 0$ and $-\alpha < \beta < \alpha$ then the steady state $\bar{\mathbf{y}} = \mathbf{0}$ is unique is unstable (see panel (c) in

Figure 4.13);

4. if $\alpha = \beta$ then there will be an infinite number of steady states along a line $y_1 + y_2 = 0$, that is there is a non-empty center manifold $\mathcal{W}^c = \{ \mathbf{y} \in \mathrm{Y} : y_1 + y_2 = 0\}$. Furthermore, if $\alpha < 0$ ($\alpha > 0$) then the phase diagram is qualitatively similar to a stable (unstable) saddle-node, with the trajectories converging to (diverging from) $\mathcal{W}^c$;

5. if $\alpha = -\beta$ then there will be an infinite number of steady states along a line $y_1 - y_2 = 0$, that is there is a non-empty center manifold $\mathcal{W}^c = \{ \mathbf{y} \in \mathrm{Y} : y_1 - y_2 = 0\}$. Furthermore, if $\alpha < 0$ ($\alpha > 0$) then the phase diagram is qualitatively similar to a stable (unstable) saddle-node, with the trajectories converging to (diverging from) $\mathcal{W}^c$.

If $\mathbf{B} \neq \mathbf{0}$ the phase diagrams for the three first cases are the same with the exception that the steady state is different from the origin (see equation (4.32)). The last two cases differ: if $\mathbf{B} \neq \mathbf{0}$ and $|\alpha| = |\beta|$ then there will be no steady states.

4.3 Algebraic characterization of the solutions of planar ODE

When time is the independent variable, we can a complete characterization of the behavior of the solution over time, and how it depends on the parameters. There is a general characterization in applied mathematics and a particular use in economics.

Applied mathematics offers two main types of approaches: **stability analysis**, when we consider the parameters of the model fixed and study the long run behavior of its solution, or **bifurcation analysis** when a change in the parameters leads to several different types of phase diagrams. In economics, we have **comparative dynamics analysis** when we change locally the value of parameters without changing the qualitative characterization of the dynamics, or its type of phase diagram

In stability analysis we are concerned with the behavior of the solution by highlighting the **order relationship** within the space of the independent variable when the interval of time evolves. Typically, $t = 0$ refers to the present moment and $t = \infty$ to the very long future (or, in some cases, to a state in which time becomes irrelevant). Two perspectives are possible: a forward perspective when we want to project into the future a state of a system, for instance $\mathbf{y}(0) = \mathbf{y}_0$ with $\mathbf{y}_0$ a known element of Y, which we know now; or a backward perspective, when we fix a state in the future, for instance $\mathbf{y}(\infty) = \mathbf{y}$, and want to know which solutions would lead to it.

4.3.1 Solution of planar linear equations, with time as the independent variable

In this subsection we provide a bird's eye view on the solution of a planar linear autonomous ODE, recalling our results from chapter 3, and expanding it by studying the dynamics of the solutions.

Consider the planar linear ODE (4.1), with matrices given in equation (4.2).

Matrix A and the associated phase diagram

In chapter 3 we saw matrix $\mathbf{A}$, can be of the two types:

First, recalling that the eigenvalues of matrix $\mathbf{A}$ are the numbers

$$\lambda_{\mp} = \frac{\text{trace}(\mathbf{A})}{2} \mp \sqrt{\Delta(\mathbf{A})}, \text{ where } \Delta(\mathbf{A}) = \left(\frac{\text{trace}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A}),$$

then matrix $\mathbf{A}$ is similar to one of the Jordan canonical forms

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}, \ \mathbf{\Lambda}_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \text{ or } \mathbf{\Lambda}_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

where all the parameters are real numbers, if $\Delta(\mathbf{A}) > 0$, $\Delta(\mathbf{A}) = 0$, or $\Delta(\mathbf{A}) < 0$, respectively. Furthermore, $\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$ where $\mathbf{P}$ is the (non-singular) eigenvector matrix. Second, matrix $\mathbf{A}$ is non-canonical if it takes one of the following two forms

$$\mathbf{\Lambda}_d = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \text{ or } \mathbf{\Lambda}_h = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix},$$

and if $\Delta(\mathbf{A}) > 0$ the two eigenvalues are real and distinct, and satisfy $\lambda_+ > \lambda_-$, and if $\Delta(\mathbf{A}) = 0$ they are equal, and real $\lambda_+ = \lambda_- = \lambda$, and if $\Delta(\mathbf{A}) = 0$ they are complex conjugate $\lambda_{\pm} = \alpha \pm \beta\, i$, where $i = \sqrt{-1}$.

From the results of the last section we know that: if $\det(\mathbf{A}) > 0$ and $\Delta(\mathbf{A}) > 0$ the phase diagram is a node; if $\det(\mathbf{A}) > 0$ and $\Delta(\mathbf{A}) < 0$ the phase diagram is a focus, and if $\det(\mathbf{A}) < 0$ it is a saddle.

Furthermore, we know that the eigenvector matrix determines the eigenspaces, and the dimension and slopes of the stable and unstable manifolds. Recall that if $\Delta(\mathbf{A}) \neq 0$, the eigenvector matrix concatenates the eigenvectors associated to the two eigenvalues λ_- and λ_+, is

$$\mathbf{P} = \mathbf{P}^- \,|\, \mathbf{P}^+ \equiv \begin{pmatrix} P_1^- & P_1^+ \\ P_2^- & P_2^+ \end{pmatrix},$$

and if $\Delta(\mathbf{A}) = 0$ the eigenvector matrix concatenates a simple and a generalized eigenvector $\mathbf{P} = \mathbf{P}^s|\mathbf{P}^g$ (see Appendix to chapter 3).

4.3.2 Steady states

Definition 1 (Steady state). *A steady state is an element of* Y *belonging to the set*

$$\bar{\mathbf{y}} = \left\{\mathbf{y} \in \mathrm{Y} : \ \mathbf{A}\,\mathbf{y} + \mathbf{B} = \mathbf{0}\right\}.$$

Proposition 5 (Existence and number of fixed points). *Let the set of steady states be given in definition 1:*

1. *If* $\det(\mathbf{A}) \neq 0$*, that is, if all eigenvalues of* $\mathbf{A}$ *are different from zero, then there is an unique steady, and it is given by*
$$\bar{\mathbf{y}} = -\mathbf{A}^{-1}\,\mathbf{B}.$$

2. *If* $\det(\mathbf{A}) = 0$ *and* $\text{trace}(\mathbf{A}) \neq 0$ *then the two eigenvalues are real, distinct, there is one eigenvalue which is equal to zero. Two cases are possible:*
   (a) if $\text{trace}(\mathbf{A}) < 0$ *then* $\lambda_+ = 0 > \lambda_-$*, and* $P_2^+ b_2 = P_1^+ b_1$*, there is an infinite number of steady states over the one-dimensional manifold (a line)*
$$\bar{\mathbf{y}} \in \{\ (y_1, y_2) \in \mathrm{Y} :\ P_1^-(\lambda_- y_2 - b_2) = P_2^-(\lambda_+ y_1 - b_1)\},$$

   (b) if $\text{trace}(\mathbf{A}) > 0$ *then* $\lambda_+ > 0 = \lambda_-$*, and* $P_1^- b_2 = P_2^- b_1$*, and there is an infinite number of steady states over the one-dimensional manifold*
$$\bar{\mathbf{y}} \in \{\ (y_1, y_2) \in \mathrm{Y} :\ P_2^+(\lambda_+ y_1 - b_1) = P_1^+(\lambda_- y_2 - b_2)\}\ .$$

3. *if* $\mathbf{A} = \mathbf{0}$ *and* $P_2^+ b_2 - P_1^+ b_1 = P_1^- b_2 - P_2^- b_1 = 0$ *then we have an infinity of equilibrium points belonging to a two-dimensional manifold (i.e.,* $\bar{\mathbf{y}} = \mathrm{Y}$*).*

4. *If* $\Delta(\mathbf{A}) = \text{trace}(\mathbf{A}) = 0$*, but the Jordan canonical matrix of* $\mathbf{A}$ *is of type* $\mathbf{\Lambda}_2$*, then there are two equal eigenvalues,* $\lambda = 0$*, and if* $P_2^g b_1 = P_1^g b_2$ *then there is an infinite number of equilibrium points belonging to a one-dimensional manifold, whose coefficients is given by the simple eigenvalue*
$$\bar{\mathbf{y}} \in \{\ (y_1, y_2) \in \mathrm{Y} :\ P_2^s(y_1 - b_1) = P_1^s(y_2 - b_2)\}.$$

5. *If none of the former conditions hold there are no steady states.*

Proof. A steady state is a point $\mathbf{y}$ such that $\mathbf{A}\mathbf{y} = -\mathbf{B}$. If $\det(\mathbf{A}) \neq 0$ then a there is a unique inverse matrix $\mathbf{A}^{-1}$ and therefore a unique fixed point exits $\bar{\mathbf{y}} = -\mathbf{A}^{-1}\mathbf{B}$. If matrix $\mathbf{A}$ is singular, that is $\det(\mathbf{A}) = 0$, then a classical inverse does not exist. In this case, observe that $\mathbf{A}\mathbf{y} = -\mathbf{B}$ is equivalent to $\mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}\mathbf{y} = -\mathbf{B}$ and also $\mathbf{\Lambda}\mathbf{P}^{-1}\mathbf{y} = -\mathbf{P}^{-1}\ \mathbf{B}$. Because in this case there are only real eigenvalues, the expansion of this equation can take several forms. If $\Delta(\mathbf{A}) > 0$ we can expand $\mathbf{\Lambda}\mathbf{P}^{-1}\mathbf{y} = -\mathbf{P}^{-1}\ \mathbf{B}$ as
$$\begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \begin{pmatrix} P_2^+ & -P_1^+ \\ -P_2^- & P_1^- \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} P_2^+ & -P_1^+ \\ -P_2^- & P_1^- \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$
Then: (1) if $\lambda_+ = 0 > \lambda_-$ then
$$P_2^+ b_2 = P_1^+ b_1, \text{ and } P_1^-(\lambda_- y_2 - b_2) = P_2^-(\lambda_- y_1 - b_1);$$
(2) if $\lambda_+ > 0 = \lambda_-$ then
$$P_1^- b_2 = P_2^- b_1, \text{ and } P_2^+(\lambda_+ y_1 - b_1) = P_1^+(\lambda_+ y_2 - b_2);$$

or (3) if $\lambda_+ = \lambda_- = 0$, then the Jordan canonical form is $\mathbf{\Lambda}_1 = \mathbf{0}$ if and only if $\mathbf{A} = \mathbf{0}$, the expansion is $P_2^+ b_2 - P_1^+ b_1 = P_1^- b_2 - P_2^- b_1 = 0$. At last, if there $\Delta(\mathbf{A}) = 0$ and the Jacobian matrix is $\mathbf{\Lambda}_2$ with $\lambda = 0$, steady states exist if and only if

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_2^g & -P_1^g \\ -P_2^s & P_1^s \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} P_2^g & -P_1^g \\ -P_2^s & P_1^s \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

which is equivalent to

$$P_2^s b_1 = P_1^g b_2, \text{ and } P_2^s(y_1 - b_1) = P_1^g(y_2 - b_2)$$

In all other cases, fixed points will not exist. □

Table 4.2 lists the previous results.

Table 4.2: Number of steady states

	$\det(\mathbf{A}) \neq 0$	$\det(\mathbf{A}) = 0$	
		$\text{trace}(\mathbf{A}) \neq 0$	$\text{trace}(\mathbf{A}) = 0$
$\mathbf{B} = \mathbf{0}$	unique	infinite (co dim 1)	infinite (co-dim 2)
$\mathbf{B} \neq \mathbf{0}$		zero	infinite (co-dim 1)

If the initial point is a steady state $\mathbf{y}(0) = \overline{\mathbf{y}}$ the solution is stationary. If $\mathbf{y}(0) \neq \overline{\mathbf{y}}$ the solution is time independent. The time dependency of solutions can be studied from the point of view of their stability properties and from their recurrence properties.

4.3.3 Stability analysis

In this section we characterize the trajectories generated by a planar ODE, $\Big(\mathbf{y}(t)\Big)_{t\in[0,\infty}$ regarding their convergence properties.

Definition 2 (Stability definitions)**.**

A solution is ***asymptotically stable*** *if, for for an arbitrary* $\mathbf{y}(0)$ *in a neighborhood of* $\overline{\mathbf{y}}$*, it converges asymptotically to* $\overline{\mathbf{y}}$*: i.e.,* $\lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{w}}$ *for* $|\mathbf{y}(0) - \overline{\mathbf{y}}| < \epsilon$ *for a given* ϵ*.*

A solution is ***stable*** *if, for an* $\mathbf{y}(0)$ *in a neighborhood of* $\overline{\mathbf{y}}$*, the solution stays close to* $\overline{\mathbf{y}}$*, for every* $t \in (0, \infty)$ *but does not converges asymptotically to* $\overline{\mathbf{y}}$*.*

A solution is ***unstable*** *if, for an* $\mathbf{y}(0)$ *in a neighborhood of* $\overline{\mathbf{y}}$*, the solution becomes asymptotically unbounded, i.e.,* $\lim_{t\to\infty} \mathbf{y}(t) = \pm\infty$*.*

*A solution is **conditionally stable** if, for a particular values* $\mathbf{y}(0)$*, say* $\mathbf{y}^s(0)$*, in a neighborhood of* $\overline{\mathbf{y}}$*, the solution converges asymptotically to* $\overline{\mathbf{y}}$*, but a small deviation from* $\mathbf{y}^s(0)$ *turns the solution unstable.*

In order to study the stability of the solutions of ODE (4.1), we start with the cases in which there is a unique steady state.

The following result is useful:

Lemma 12 (Representation of the solution)**.** *Consider the planar ode* (4.1)*, and assume that* $\det(\mathbf{A}) \neq 0$*. Then there is a unique steady state* $\overline{\mathbf{y}} \in \mathrm{Y}$ *and the solution of the ODE can be equivalently written as*

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\,\mathbf{e}^{\boldsymbol{\Lambda}\,\mathbf{t}}\;\mathbf{w}(0) \tag{4.33}$$

where $\mathbf{w}(0) = \mathbf{P}^{-1}\;(\mathbf{y}(0) - \overline{\mathbf{y}})$ *is a function of the an arbitrary point* $\mathbf{y}(0) \in \mathrm{Y}$*.*

Proof. Let the steady state be $\overline{\mathbf{y}}$. Introduce the transformation $\mathbf{y}(t) - \overline{\mathbf{y}} = \mathbf{P}\mathbf{w}(t)$. Then $\mathbf{w}\,(t) = \mathbf{P}^{-1}(\mathbf{y}(t) - \overline{\mathbf{y}})$ and $\dot{\mathbf{w}} = \mathbf{P}^{-1}\dot{\mathbf{y}} = \mathbf{P}^{-1}\,(\mathbf{A}\mathbf{y}+\mathbf{B}) = \mathbf{P}^{-1}\Big(\mathbf{A}\,(\mathbf{P}\mathbf{w}+\overline{\mathbf{y}})+\mathbf{B}\Big) = \boldsymbol{\Lambda}\mathbf{w}+\mathbf{P}^{-1}\mathbf{A}\overline{\mathbf{y}}\;+\mathbf{P}^{-1}\mathbf{B} = \boldsymbol{\Lambda}\mathbf{w}-\mathbf{P}^{-1}\mathbf{B}+\mathbf{P}^{-1}\mathbf{B} = \boldsymbol{\Lambda}\mathbf{w}$ for any matrix $\boldsymbol{\Lambda}$. Then, we get equivalently $\dot{\mathbf{w}} = \boldsymbol{\Lambda}\mathbf{w}$, which has solution $\mathbf{w}(t) = \mathbf{e}^{\boldsymbol{\Lambda}\mathbf{t}}\,\mathbf{w}(0)$, where $\mathbf{w}(0)$ is, in the original variable given by $\mathbf{w}(0) = \mathbf{P}^{-1}\,(\mathbf{y}(0) - \overline{\mathbf{y}})$. □

The eigenvalues of $\mathbf{A}$ not only determine the number of steady states but also their stability properties:

Proposition 6. *The asymptotic dynamic characteristics of the solution of equation* (4.1) *is determined by the real part of the eigenvalues of matrix* $\mathbf{A}$*:*

1. *if all the eigenvalues have negative real parts then all solutions of the ODE are asymptotically stable;*
2. *if all eigenvalues have positive real parts then all solutions are unstable;*
3. *if there is one negative and one positive eigenvalue then the solution is conditionally stable: it is unstable if* $w_1(0) = 0$ *and it is asymptotically stable if* $w_2(0) = 0$*;*
4. *if the eigenvalues are complex with zero real part the solution is stable but not asymptotically stable;*
5. *if there is one zero eigenvalue the fixed point is a one-dimensional manifold (a center manifold), the solution will converge to it if the other eigenvalue is negative (i.e., in case* $\lambda_+ = 0$ *and* $\lambda_- < 0$*) and will not converge to it if the other eigenvalue is positive (i.e., in case* $\lambda_+ > 0$ *and* $\lambda_- = 0$*).*

Proof. Consider figure 3.1 in chapter 3. The solution of the ODE (4.1) can take one of the following three forms: First, if

1. if $\Delta(\mathbf{A}) > 0$, the general solution is

$$\mathbf{y}(t) = \overline{\mathbf{y}} + w_1(0)\,\mathbf{P}^-\,e^{\lambda_- t} + w_2(0)\,\mathbf{P}^+\,e^{\lambda_+ t};$$

   or, equivalently

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + w_1(0) \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} e^{\lambda_-\,t} + w_2(0) \begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix} e^{\lambda_+\,t}.$$

   Then, letting $\mathbf{w}(0) \neq \overline{\mathbf{w}}$: (1) the solution is asymptotically state if $0 > \lambda_+ > \lambda_-$; (2) it is conditionally stable if $\lambda_- < 0 < \lambda_+$ and $w_2(0) = 0$; and (3) it is unstable if $0 > \lambda_+ > \lambda_- > 0$;

2. if $\Delta(\mathbf{A}) = 0$, the general solution is

$$\mathbf{y}(t) = \overline{\mathbf{y}} + e^{\lambda t}\left(\mathbf{P}^s(w_1(0) + w_2(0)\,t) + w_2(0)\,\mathbf{P}^g\right)$$

   or, equivalently

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + e^{\lambda t}\left((w_1(0) + w_2(0)t) \begin{pmatrix} P_1^s \\ P_2^s \end{pmatrix} + w_2(0) \begin{pmatrix} P_1^g \\ P_2^g \end{pmatrix}\right).$$

   Then, letting $\mathbf{w}(0) \neq \overline{\mathbf{w}}$: (1) the solution is asymptotically state if $\lambda < 0$; or (2) it is unstable if $\lambda > 0$;

3. if $\Delta(\mathbf{A}) < 0$, the general solution is

$$\begin{aligned} \mathbf{y}(t) &= \overline{\mathbf{y}} + e^{\alpha t}\left((w_1(0)\cos\beta t + w_2(0)\sin\beta t)\mathbf{P}^1 + (w_2(0)\cos\beta t - w_1(0)\sin\beta t)\mathbf{P}^2\right) = \\ &= \overline{\mathbf{y}} + e^{\alpha t}\left(w_1(0)(\cos\beta t\mathbf{P}^1 - \sin\beta t\mathbf{P}^2) + w_2(0)(\sin\beta t\mathbf{P}^1 + \cos\beta t\mathbf{P}^2)\right). \end{aligned}$$

   or, equivalently,

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + e^{\alpha t}\left(w_1(0) \begin{pmatrix} P_1^- \cos\beta t - P_1^+ \sin\beta t \\ P_2^- \cos\beta t - P_2^+ \sin\beta t \end{pmatrix} + w_2(0) \begin{pmatrix} P_1^- \sin\beta t + P_1^+ \cos\beta t \\ P_2^- \sin\beta t + P_2^+ \cos\beta t \end{pmatrix}\right).$$

   Then, letting $\mathbf{w}(0) \neq \overline{\mathbf{w}}$: (1) the solution is asymptotically state if $\alpha < 0$; (2) it is unstable if $\alpha > 0$; or (3) it is stable but non conditionally stable if $\alpha = 0$. In the last case the solution is periodic.

□

The dynamic behavior of the solution to equation (4.1) from the perspective of its stability properties is:

Theorem 1 (Stability properties for planar linear ODE)**.** *Consider the planar ODE* (4.1)*. Assume that a fixed point* $\overline{\mathbf{y}} \in Y$ *exists if* $\det(\mathbf{A}) \neq 0$ *or that an infinite number of fixed points exist if* $\det(\mathbf{A}) = 0$*. The asymptotic properties of the solution as a function of the trace and determinant of* $\mathbf{A}$ *are:*

1. *asymptotic stability if and only if* $\text{trace}(\mathbf{A}) < 0$ *and* $\det(\mathbf{A}) \geq 0$*;*
2. *saddle path (or conditional) stability if and only if* $\det(\mathbf{A}) < 0$*;*
3. *instability if and only if* $\text{trace}(\mathbf{A}) > 0$ *and* $\det(\mathbf{A}) \geq 0$*;*
4. *stability but not asymptotic stability if* $\text{trace}(\mathbf{A}) = 0$ *and* $\det(\mathbf{A}) \geq 0$.

Table 4.3 tabulates the content of theorem 1.

Table 4.3: Stability of steady states

	$\det(\mathbf{A}) < 0$	$\det(\mathbf{A}) = 0$	$\det(\mathbf{A}) > 0$
$\text{trace}(\mathbf{A}) < 0$		asymptotically stable	asymptotically stable
$\text{trace}(\mathbf{A}) = 0$	conditionally stable	stationary solutions	stable
$\text{trace}(\mathbf{A}) > 0$		unstable	unstable

4.3.4 Partition of the space Y

Assuming that the initial arbitrary value $\mathbf{y}(0) \neq \bar{\mathbf{y}}$, we just saw that the solution has three types of behavior: it converges asymptotically to a steady state, it diverges over time or it has a limited amplitude, neither converging nor diverging by much. This allows for a partition of set Y into three invariant subsets (which can be empty or not) such that a solution of the ODE will stay in one of them for the whole adjustment between $t = 0$ and $t = \infty$.

The **attracting set** or **stable manifold** as the subset of point such that solutions converge to an equilibrium point

$$\mathcal{W}^s = \left\{ \mathbf{y}(0) \in \mathrm{Y} : \lim_{t \to \infty} \mathbf{y}(t; \mathbf{y}(0)) = \bar{\mathbf{y}} \right\}$$

the **repelling set** or **unstable manifold** as the subset of point such that solutions become asymptotically unbounded

$$\mathcal{W}^u = \left\{ \mathbf{y}(0) \in \mathrm{Y} : \lim_{t \to \infty} \mathbf{y}(t; \mathbf{y}(0)) = \pm\infty \right\}$$

and the **center manifold**, denoted by $\mathcal{W}^c$, as the subset of points which are neither asymptotically stable nor unstable.

This introduces a partition over the state space Y:

$$\mathrm{Y} = \mathcal{W}^s \oplus \mathcal{W}^u \oplus \mathcal{W}^c.$$

In the case of a linear ODE the **stable**, **unstable**, and **center** manifolds are **global** manifolds because there is a unique steady state or a unique center manifold. As we will see, in the case of non-linear ODE's, which can have more than one, but finite in number, steady states, we distinguish between **local manifolds** and **global manifolds** when they refer to a particular steady state or to the whole space.

4.3.5 Eigenspaces and stability analysis

The solution of an ODE (4.1) satisfy, in most cases, a superposition principle, because it is a weighted function of two exponential functions. For example, if $\Delta(\mathbf{A}) > 0$ we saw that the solution can be written as

$$\mathbf{y}\ (t) = \overline{\mathbf{y}}\ + w_1(0)\mathbf{P}^-\ e^{\lambda_- t} + w_2(0)\,\mathbf{P}^+\ e^{\lambda_+ t}.$$

That is, the solution of the ODE is a superposition of two elementary function $e^{\lambda_- t}$ and $e^{\lambda_+ t}$, acting on the directions defined by the eigenvector $\mathbf{P}^-$ and $\mathbf{P}^+$, and weighed by $\mathbf{w}(0)$ which is a function of the arbitrary value $\mathbf{y}(0) \in \mathrm{Y}$. In other words, the elementary components of the time behavior of the solutions, $e^{\lambda_+ t}$ and $e^{\lambda_- t}$, are linearly transformed by the eigenvectors $\mathbf{P}^1$ and $\mathbf{P}^2$.

We have defined the **eigenspaces** as the subsets of space Y which are travelled by those two elementary solutions:

$$\begin{aligned} \mathcal{E}^- &= \{\ \mathbf{y} \in \mathrm{Y} :\ \text{spanned by } \mathbf{P}^-\} \\ \mathcal{E}^+ &= \{\ \mathbf{y}\ \in \mathrm{Y} :\ \text{spanned by } \mathbf{P}^+\} \end{aligned}$$

Clearly the range of $\mathbf{y}$ is spanned by those two eigenvectors: i.e., $\mathrm{Y} = \mathcal{E}^- \oplus \mathcal{E}^+$.

If if $\Delta(\mathbf{A}) > 0$ we can determine again the eigenspaces by making $w_2(0) = 0$ and $w_1(0) = 0$, respectively, [8] yielding

$$\mathcal{E}^- = \{\mathbf{y} \in \mathrm{Y} :\ P_1^-(y_2 - \overline{y}_2) = P_2^-(y_1 - \overline{y}_1)\}$$

and

$$\mathcal{E}^+ = \{\mathbf{y} \in \mathrm{Y} :\ P_1^+(y_2 - \overline{y}_2) = P_2^+(y_1 - \overline{y}_1)\}$$

The stable, unstable and center manifolds, are the global stable, unstable and center manifolds which partition set Y, according to the dynamic properties of the solution to a linear ODE. They are, therefore spanned by the eigenspaces associated to the eigenvalues with negative, positive and zero real parts. Formally the **stable manifold** is spanned by the eigenspaces which are associated to the eigenvectors with negative real parts

$$\mathcal{W}^s \equiv \oplus_{j\in\pm}\ \{\ \mathcal{E}^j : \mathrm{Re}(\lambda_j) < 0\},$$

the **unstable manifold** is spanned by the eigenspaces which are associated to the eigenvectors with positive real parts

$$\mathcal{W}^u \equiv \oplus_{j\in\pm}\{\ \mathcal{E}^j : \mathrm{Re}(\lambda_j) > 0\},$$

and the **center manifold** is spanned by the eigenspaces which are associated to the eigenvectors with zero real parts

$$\mathcal{W}^c \equiv \oplus_{j\in\pm}\{\ \mathcal{E}^j : \mathrm{Re}(\lambda_j) = 0\}.$$

[8] We can determine the eigenvector $\mathcal{E}^-$ if we set $w_2(0) = 0$ we have $w_1(0)e^{\lambda_- t}P_1^- = y_1(t) - \overline{y}_1$ and $w_1(0)e^{\lambda_- t}P_2^- = y_2(t) - \overline{y}_2$. Thus $w_1(0)e^{\lambda_- t} = \dfrac{y_1(t) - \overline{y}_1}{P_1^-} = \dfrac{y_2(t) - \overline{y}_2}{P_2^-}$. We proceed in an analogous way for $\mathcal{E}^+$.

Again we have

$$\mathcal{W}^s \oplus \mathcal{W}^u \oplus \mathcal{W}^c = \mathrm{Y}.$$

Let n_-, n_+ and n_c be respectively the number of eigenvalues with negative, positive and zero real parts. Another way to see the relationship between the eigenspaces and the range of the dynamical system is based on the observation that

$$n_- + n_+ + n_c = 2.$$

and that the dimension of the there eigenspaces are therefore

$$\dim(\mathcal{W}^s) = n_-,\ \dim(\mathcal{W}^u) = n_+,\ \dim(\mathcal{W}^c) = n_c,$$

implying

$$\dim(\mathcal{W}^s) + \dim(\mathcal{W}^u) + \dim(\mathcal{W}^c) = \dim(\mathrm{Y}) = 2.$$

Therefore, for a planar ODE we have:

1. if all eigenvalues have negative real parts, i.e., if $n_- = 2$, then $\mathcal{W}^s = \mathcal{E}^- \oplus \mathcal{E}^+ = \mathrm{Y}$, and $\mathcal{W}^u$ and $\mathcal{W}^c$ are empty, which means that $\mathcal{W}^s$ is spanned by $\mathcal{E}^-$ and $\mathcal{E}^+$ (i.e, the elements in $\mathcal{W}^s$ are a weighted sum of elements of $\mathcal{E}^-$ and $\mathcal{E}^+$). Then Y is the **attracting set**;

2. if all eigenvalues have positive real parts, i.e., if $n_+ = 2$, then $\mathcal{W}^u = \mathcal{E}^- \oplus \mathcal{E}^+ = \mathrm{Y}$, and $\mathcal{W}^s$ and $\mathcal{W}^c$ are empty. Then Y is the **repelling set**;

3. if there is a saddle point, i.e., if $n_- = n_+ = 1$, then $\mathcal{W}^s = \mathcal{E}^-$, $\mathcal{W}^u = \mathcal{E}^+$ and , and $\mathcal{W}^c$ is empty. Then $\mathcal{W}^s$ is the **attracting set** and $\mathcal{W}^u$ is the **repelling set**;

4. if there is at least one eigenvalue with zero real part, i.e., if $n^c \in \{1, 2\}$, then $\mathcal{W}^c$ is non-empty. Three cases are possible (see the proof of Proposition 5):

   (a) first, if $\lambda_- < 0 = \lambda_+$ then $\mathcal{W}^c = \{\mathbf{y} \in \mathrm{Y} :\ P_1^-(\lambda_- y_2 - b_2) = P_2^-(\lambda_- y_1 - b_1)\}$, $\mathcal{W}^s = \mathrm{Y}/\mathcal{W}^c$ and $\mathcal{W}^u$ is empty;

   (b) second, if $\lambda_- = 0 < \lambda_+$ then $\mathcal{W}^c = \{\mathbf{y} \in \mathrm{Y} :\ P_2^+(\lambda_+ y_1 - b_1) = P_1^+(\lambda_+ y_2 - b_2)$, and $\mathcal{W}^u = \mathrm{Y}/\mathcal{W}^c$ and $\mathcal{W}^s$ is empty,

   (c) third, $\mathcal{W}^c = \mathrm{Y}$ and $\mathcal{W}^s$ and $\mathcal{W}^u$ are both empty if there are two eigenvalues with zero real parts.

4.3.6 Recurrence of solutions

We can classify solutions regarding their time profile into stationary, non-stationary, monotonic, oscillatory, periodic solutions and hump-shaped. We use our previous transformation $\mathbf{y}(t) - \overline{\mathbf{y}} = \mathbf{P}\,\mathbf{w}(t)$, because, the main dynamic characteristics of the solution are generated by $\dot{\mathbf{w}}$.

Stationary solutions We say the solution is stationary if $\mathbf{y}(t) = \bar{\mathbf{y}}$ is a constant for all $t \in \mathrm{T}$. In this case $\dot{\mathbf{w}}(t) = \mathbf{0}$ for all t and we already saw under which circumstances solutions are stationary. We can say that a solution is **asymptotically stationary** if it converges asymptotically to a steady state, i.e, $\lim_{t\to\infty} \dot{\mathbf{y}}(t) = \mathbf{0}$ or $\lim_{t\to\infty} \mathbf{y}(t) = \bar{\mathbf{y}}$.

Non-stationary solutions We could say that a solution is non-stationary if $\mathbf{y}(t) \neq \bar{\mathbf{y}}$ for some $t \in \mathrm{T}$. However, this designation is commonly reserved to solutions which are not asymptotically stationary that is to solutions of time $\lim_{t\to\infty} \dot{\mathbf{w}}(t) \neq 0$ or $\lim_{t\to\infty} \mathbf{y}(t) = \bar{\mathbf{y}}(t) = \pm\infty$. Solution are also non-stationary if a steady state does not exist (see Table 4.2).

Monotonic solutions We say the solution is monotonic if $\mathrm{sign}(\dot{\mathbf{w}}(t))$ is the same for all $t \in \mathrm{T}$. This means that the solution is monotonically increasing if $\dot{\mathbf{w}}(t) > \mathbf{0}$ for all t, it is monotonically decreasing if $\dot{\mathbf{w}}(t) < \mathbf{0}$ for all t. A stationary solution can be seen as a particular type of monotonic solution.

Oscillatory solutions A solution is oscillatory if $\mathbf{w}(t) = \mathbf{w}(t+p(t))$ for $t \in \mathrm{T}$ and time-dependent period $p(t) \in \mathrm{T}$: the solution is repeated in increasing intervals if $p^{'}(t) > 0$ or in decreasing intervals if $p^{'}(t) < 0$. For these solutions, there is a sequence of points, increasing or decreasing over time $\tau \in \{t_0, t_1, \dots, t_s, \dots\}$ such that $\dot{\mathbf{w}}(\tau) = 0$. In our case if there are two complex eigenvalues with non-zero real part, that is $\alpha \neq 0$, then the solution is oscillatory

$$\mathbf{w}(t) = e^{\alpha t} \begin{pmatrix} w_1(0)\cos\beta t + w_2(0)\sin\beta t \\ w_2(0)\cos\beta t - w_1(0)\sin\beta t \end{pmatrix}.$$

Periodic solutions If a solution satisfies $\mathbf{w}(t) = \mathbf{w}(t+p)$ for $t \in \mathrm{T}$ and $p \in T$ it is a periodic solution period p. This is a particular case of an oscillatory solution in which the period is constant. In our case if there are two complex eigenvalues with zero real part then the solution is periodic

$$\mathbf{w}(t) = \begin{pmatrix} w_1(0)\cos\beta t + w_2(0)\sin\beta t \\ w_2(0)\cos\beta t - w_1(0)\sin\beta t \end{pmatrix}.$$

This case occurs if and only if $\mathrm{trace}(\mathbf{A}) = 2\alpha = 0$. Observe that in this case and if we transform the system into polar coordinates (see section 4.A.1 in the appendix) we have $r(t) = r_0$ constant and $\theta(t) = \theta_0 - \beta t$.

Hump-shaped solutions If the solution of a planar equation is such that only one variable satisfies $\dot{y}_i(t) = 0$ for a finite $t \in \mathrm{T}$ and the other variable y_{-i} is monotonic, then we say the solution is hump-shaped. This case only occurs for the general homogeneous equation when there are eigenvalues with real parts. Differently from oscillatory trajectories, there only one value of time such that $\dot{y}_i(t) = 0$.

4.4 Bifurcation analysis

In applied modelling, ODEs depend on parameters. That is, we are interested in models of type [9]

$$\dot{\mathbf{y}} = F(\mathbf{y}, \varphi) = \mathbf{A}(\varphi)\,\mathbf{y} + \mathbf{B}(\varphi) \tag{4.34}$$

where φ is a parameter or a vector of m parameters with domain in a set Φ, that is $\varphi \in \Phi \subset \mathbb{R}^m$. This implies that the solution of the ODE is a mapping $\mathbf{y} : \mathrm{T} \times \Phi \to \mathrm{Y} \subseteq \mathbb{R}^2$.

According to our previous study on the dynamics of the planar ODE we saw that the most relevant characteristics of the dynamics are related with stability or instability of the solution, and with the monotonous or oscillatory nature of its path. These properties tend to be generic, in the sense that they can be satisfied for a wide change in the elements of $\mathbf{A}$, and they change by passing through non-generic cases, that is cases in which a small change in an element of $\mathbf{A}$ triggers a change in the phase diagram.

Bifurcation analysis studies the qualitative changes in the dynamics of the solution of the ODE (4.34) for variation of parameters within the set Φ. In other words, it studies which types of phase diagrams can occur. This is done by finding bifurcations: that is by identifying parameters which when they change by passing through specific critical values there will be a qualitative change in the phase diagram. From our previous results this is tantamount to finding changes in the eigenvalues of matrix $\mathbf{A}$, [10], that is, changes in the trace and the determinant of $\mathbf{A}$.[11]

Bifurcation analysis is tantamount to finding a partition in the set of the parameters space Φ which is associated to the stability properties of the model, that is, to the different dimensions of the stable, unstable and center manifolds.

Assume there is a steady state, $\overline{\mathbf{y}}(\varphi)$, which is a function of the parameters of the model. A **bifurcation** occurs for a value of the parameter $\varphi = \varphi^*$ such that the local dimension of the eigenspaces of $\overline{\mathbf{y}}(\varphi^*)$ change.

Our classification of the phase diagrams in Table 4.1 allows us to classify bifurcations according to the number of parameters that should change in order to see that the changes in the stability occurs when there are eigenvalues with zero real part.

Let us define

$$T(\varphi) = \mathrm{trace}(\mathbf{A}(\varphi)), \text{ and } D(\varphi) = \det(\mathbf{A}(\varphi)).$$

We can also define a function for the discriminant $\Delta(\varphi) = \left(\frac{T(\varphi)}{2}\right)^2 - D(\varphi)$.

The **co-dimension** of a bifurcation refers to the number of parameters which need to change to bring about a bifurcation.

In planar ODEs there are only bifurcations of co-dimension one and two. Bifurcation of co-dimension one occur if there is $\varphi = \varphi^\star$ such that $D(\varphi^*) = 0$, that is if there is a parameter value

[9] Sometimes called exogenous variables in economic models.

[10] We will generalize this approach for non-linear ODEs in next chapters.

[11] Observe that in the scalar ODE we only needed a parameter to characterize the stability properties of the ODE. The trace and the determinant are the extension of the coefficient of y to the planar case.

such that there is a zero eigenvalue, and bifurcation of co-dimension two occur if there is $\varphi = \varphi^\star$ such that $T(\varphi^*) = 0$ and $D(\varphi^*) > 0$ that is if there is a parameter value such that there is a complex eigenvalue with zero real part.

We can determine co-dimension one bifurcations by solving

$$\begin{cases} \mathbf{A}(\varphi)\,\mathbf{y} + \mathbf{B}(\varphi) = 0 \\ D(\varphi) = \det\left(\mathbf{A}(\varphi)\right) \;= 0 \end{cases} \tag{4.35}$$

for $(\mathbf{y}, \varphi)$. This allows us to partition set Φ into subsets of values in which we have saddles, stable nodes and foci, or unstable nodes and foci, which are in general intervals (they have dimension one), and the subset of bifurcation values (of dimension zero).

We can determine co-dimension two bifurcations by solving

$$\begin{cases} \mathbf{A}(\varphi)\,\mathbf{y} + \mathbf{B}(\varphi) = 0 \\ T(\varphi) = \text{trace}(\mathbf{A}(\varphi)) = 0 \\ D(\varphi) = \det\left(\mathbf{A}(\varphi)\right) \; > 0 \end{cases} \tag{4.36}$$

for $(\mathbf{y}, \varphi_1, \varphi_2)$.

We can represent geometrically the bifurcation scenarios by plotting a **bifurcation diagram**. There are two approaches for representing bifurcation diagrams.

1. By representing the partition of the Φ space, if there are at least two parameters. In this space we represent the lines $\{\varphi \in \Phi : D(\varphi) = 0\}$ and $\{\varphi \in \Phi : T(\varphi) = 0\}$, and $\{\varphi \in \Phi : \Delta(\varphi) = 0\}$,
2. By doing an implicit plot of $T(\varphi)$ and $D(\varphi)$ in the trace-determinant figure 3.1. Geometrically bifurcations exist if those lines the horizontal axis or the positive half of th evertical axis.

Example Consider the following planar ODE, where $\varphi = (\mu, b)$ is a real vector of parameters,

$$\begin{aligned} \dot{y}_1 &= \mu\, y_1 + y_2 - b \\ \dot{y}_2 &= y_2 - b. \end{aligned}$$

Assume that $b > 0$ and μ can have any sign. The coefficient matrix has trace and determinant, depending on the parameters, $T(\varphi) = 1 + \mu$ and $D(\varphi) = \mu$, and the eigenvalues are $\lambda_+(\varphi) = 1$ and $\lambda_-(\varphi) = \mu$ The bifurcation conditions in equation (4.35) take the following form

$$\begin{cases} \mu\, y_1 + y_2 - b = 0 \\ y_2 - b = 0 \\ \mu = 0. \end{cases}$$

The bifurcation point if $(\mathbf{y}, \mu) = (y_1, b, 0)$ for $\mu = 0$. If $\mu \neq 0$ there is one unique steady state $\bar{\mathbf{y}} = (0, b)$ which is a saddle point if $\mu < 0$ and an unstable node if $\mu > 0$. As det > 0 only if $\mu > 0$ then trace > 0 which implies there is no bifurcation of co-dimension two (no center).

4.5 Applications

In this section we show how to find solutions to a second-order scalar linear ODE, and to a non-autonomous linear scalar ODE by transforming them to a linear planar autonomous (first-order) ODE. This allows to a algebraic (not calculus) approach to their solution.

4.5.1 Second order linear equations

A scalar second order linear ODEs can be solved by transforming it into a planar linear ODE. Consider a general second order equation.

$$\ddot{y} - a_1\dot{y} + a_0 y + b = 0 \tag{4.37}$$

If we define $y_1 = y$ and $y_2 = \dot{y} = \dot{y}_1$, then, we can transform the equation into the system

$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = a_0 y_1 + a_1 y_2 + b$$

In matrix notation we have $\dot{\mathbf{y}} = \mathbf{A}\,\mathbf{y} + \mathbf{B}$, where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ \mathbf{A} = \begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix},\ \text{and}\ \ \mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

We readily see that $\text{trace}(\mathbf{A}) = a_1$ and $\det(\mathbf{A}) = a_1 - a_0$, and the eigenvalues are

$$\lambda_{\mp} = \frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 + a_0 - a_1}.$$

We can study the dynamics and study the qualitative dynamics by using our previous results. In particular, we see that if $a_0 \neq a_1$ there is a unique steady state and if $a_0 = a_1$ there is a steady state, for y, if $b = 0$ and there are no steady states if $b \neq 0$.

Exercise Draw the phase diagram. Draw a bifurcation diagram using the ratio a_1/a_0 as your bifurcation parameter, assuming that $b = 0$.

4.5.2 Non-autonomous scalar equations

In chapter two we studied the non-autonomous linear scalar equation $\dot{y} = \lambda\, y + \beta\, z(t)$ where $z(t) = e^{\gamma\, t}$ (equation (2.34)) where y was an endogenous variable and z was an exogenous variable, and $\lambda < 0$ and both β and γ were positive. This is a case in which we can take $z(t)$ as an additive time-dependent shock.

We could express this model as an planar forward ODE

$$\begin{aligned} \dot{y} &= \lambda\, y + \beta\, z \\ \dot{z} &= \gamma\, z \end{aligned} \tag{4.38}$$

The steady state, $(\bar{y}, \bar{z}) = (0, 0)$ is a saddle-point. The general solution is

$$\begin{aligned} y(t) &= \left(y(0) - \frac{\beta}{\gamma - \lambda}\right) e^{\lambda t} + z(0)\, \frac{\beta}{\gamma - \lambda}\, e^{\gamma t} \\ z(t) &= z(0)\, e^{\gamma t}. \end{aligned} \tag{4.39}$$

See next the solution to the initial-value problem

Exercise Draw the phase diagram.

4.6 Problems involving planar ODE's

As we saw all the solutions involve a vector of arbitrary elements of $\mathbf{y}$, $\mathbf{y}(0)$ or $\mathbf{w}(0)$. This means that we have existence but not uniqueness for **general** solutions.

In applications we introduce further information on the system. The type of **problem** involving planar ODE's depends on this additional information. We can define the following types of problems:

- if we know the initial point $\mathbf{y}(0) = \mathbf{y}_0 = (y_{1,0}, y_{2,0})$ and want to solve the problem forward in time, we say we have an **initial-value problem**;
- if we know the value of at least one variable at a point in time $T > 0$, $\mathbf{y}(T) = \mathbf{y}_T$, or $y_1(T) = y_{1,T}$, $y_2(T) = y_{2,T}$, we say we have a **boundary-value problem**;
- in economics a common problem is a mixes initial-terminal value problem, where we know the initial value for one variable and a boundary condition for the asymptotic value of another. Example: $y_1(0) = y_{1,0}$ and $\lim_{t\to\infty} e^{-\mu t} y_2(t) = 0$, where μ is a non-negative constant.

When the initial, boundary or terminal conditions are imposed we say we have **particular** solutions. Off course, the issues of existence, uniqueness and characterization still hold.

In economics it has been standard to refer to problems having an unique solution as **determinate** and to problems having multiple solutions as **indeterminate**.

We assume in the rest of this section that $\det(\mathbf{A}) \neq 0$.

4.6.1 Initial-value problems

The initial-value problem is

$$\begin{cases} \dot{\mathbf{y}} &= \mathbf{A}\,\mathbf{y} + \mathbf{B} \quad \text{for } t \in [0, \infty) \\ \mathbf{y}(0) = \mathbf{y}_0 & \quad \text{for } t = 0 \end{cases} \tag{4.40}$$

Proposition 7 (Solution to the initial-value problem). *Consider problem* (4.40) *where* $\mathbf{y}_0$ *is fixed, and assume that* $\det(\mathbf{A}) \neq 0$. *Then the solution for the initial-value problem is unique*

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P} \mathbf{e}^{\mathbf{\Lambda} \mathbf{t}} \mathbf{P}^{-1} (\mathbf{y}_0 - \overline{\mathbf{y}})$$

Proof. The general solution for a planar non-homogeneous equation is

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{t}}\ \mathbf{w}(0).$$

As $\mathbf{e}^{\mathbf{\Lambda t}}\ |_{t=0} = \mathbf{I}$ then evaluating the solution at time $t = 0$, we have

$$\mathbf{y}(0) - \overline{\mathbf{y}} = \mathbf{P}\mathbf{w}(0)$$

and because $\mathbf{P}$ is non-singular $\mathbf{w}(0) = \mathbf{P}^{-1}(\mathbf{y}(0) - \overline{\mathbf{y}})$. As the initial condition for $\mathbf{y}$ is $\mathbf{y}(0) = \mathbf{y}_0$ Plugging the initial condition we have a particular value for $\mathbf{w}(0)$

$$\mathbf{w}(0) = \mathbf{P}^{-1}(\mathbf{y}_0 - \overline{\mathbf{y}}).$$

□

4.6.2 Terminal value problems

If $T = \infty$, the terminal-value problem is

$$\begin{cases} \dot{\mathbf{y}} &= \mathbf{A}\,\mathbf{y} + \mathbf{B} \qquad \text{for } \ t \in [0, \infty) \\ \lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{y}} & \text{for } \ t \to \infty. \end{cases} \tag{4.41}$$

Proposition 8. *Consider problem* (4.41) *where* $\overline{\mathbf{y}} \in \mathrm{Y}$, *and assume that* $\det(\mathbf{A}) \neq 0$. *Then:*

(1) if $\overline{\mathbf{y}}$ *is a stable node or a stable focus then the solution is indeterminate*

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda t}}\ \mathbf{w}(0)$$

for any $\mathbf{w}(0) = \mathbf{P}^{-1}(\mathbf{y}(0) - \overline{\mathbf{y}}$ *for* $\mathbf{y}(0) \in \mathrm{Y}$*;*

(2) if $\overline{\mathbf{y}}$ *is an unstable node or an unstable focus then the solution is determinate*

$$\mathbf{y}(t) = \overline{\mathbf{y}},\ \textit{for all}\, t \in\ T$$

(3) if $\overline{\mathbf{y}}$ *is a saddle-point then the solution is indeterminate*

$$\mathbf{y}(t) = \overline{\mathbf{y}} + w_1(0)\,\mathbf{P}^{-}\,e^{\lambda_- t}.$$

for an arbitrary $w_1(0)$.

Proof. (1) If all the eigenvalues of $\mathbf{A}$ have negative real parts then

$$\lim_{t\to\infty} \mathbf{e}^{\mathbf{\Lambda t}} = \mathbf{I}_{2\times 2}$$

which implies $\lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{y}}$ independently of the value of $\mathbf{y}(0)$. (2) if all the eigenvalues of $\mathbf{A}$ have positive real parts then all the exponential functions $e^{\lambda_+ t}$, $e^{\lambda_- t}$, $e^{\lambda t}$ or $e^{\alpha t}$ become unbounded, which means that we can only have $\lim_{t\to\infty}\ \mathbf{P}\mathbf{e}^{\boldsymbol{\Lambda}\mathbf{t}}\ \mathbf{w}(0) = \mathbf{0}$ if and only if $\mathbf{w}(0) = \mathbf{0}$. Then as $\mathbf{w}(0)$ is uniquely determined, the solution is unique. (3) If the steady state is a saddle point we know that the Jacobian form of $\mathbf{A}$ is $\boldsymbol{\Lambda}_1$, the solution takes the form

$$\mathbf{y}(t) = \overline{\mathbf{y}} + w_1(0)\,\mathbf{P}^-\,e^{\lambda_- t} + w_2(0)\,\mathbf{P}^+\,e^{\lambda_+ t}$$

and $\lim_{t\to\infty} e^{\lambda_+ t} = +\infty$ and $\lim_{t\to\infty} e^{\lambda_- t} = 0$. Therefore $\lim_{t\to\infty}\mathbf{y}(t) = \overline{\mathbf{y}}$ if and only if $w_2(0) = 0$, and the solution is $\mathbf{y}(t) = \overline{\mathbf{y}} + w_1(0)\,\mathbf{P}^-\,e^{\lambda_- t}$. □

4.6.3 Initial-terminal value problems

If $T = \infty$, the initial-terminal-value problem, where we assume variable y_1 is pre-determined is

$$\begin{cases} \dot{\mathbf{y}} \ = \mathbf{A}\,\mathbf{y} + \mathbf{B}, & \text{for } \ t \in [0,\infty) \\ y_1(0) = y_{1,0}, & \text{for } \ t = 0 \\ \lim_{t\to\infty}\mathbf{y}(t) = \overline{\mathbf{y}}, & \text{for } \ t \to \infty. \end{cases} \tag{4.42}$$

Proposition 9. *Consider problem* (4.41) *where* $\overline{\mathbf{y}} \in \mathrm{Y}$*,* $y_{1,0}$ *is fixed, and assume that* $\det(\mathbf{A}) \neq 0$*. Then the solution exists and is unique*

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \frac{(y_{1,0} - \overline{y}_1)}{P_1^-}\,\mathbf{P}^-\,e^{\lambda_- t}.$$

Proof. We can take the solution of case (3) of the terminal-value problem and evaluate it at time $t = 0$ to get

$$\mathbf{y}(0) = \overline{\mathbf{y}} + w_1(0)\,\mathbf{P}^- \ \Leftrightarrow w_1(0)\,\mathbf{P}^+ + \overline{\mathbf{y}} - \mathbf{y}(0) = \mathbf{0},$$

or, expanding and substituting the initial condition

$$\begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} w_1(0) + \begin{pmatrix} \overline{y}_1 - y_{1,0} \\ \overline{y}_2 - y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As we want to solve this system for for $y_2(0) - \overline{y}_2$ and $w_1(0)$ it is convenient to re-arrange it as

$$\begin{pmatrix} P_1^- & 0 \\ P_2^- & 1 \end{pmatrix} \begin{pmatrix} w_1(0) \\ \overline{y}_2 - y_2(0) \end{pmatrix} = \begin{pmatrix} y_{1,0} - \overline{y}_1 \\ 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} w_1(0) \\ \overline{y}_2 - y_2(0) \end{pmatrix} &= \begin{pmatrix} P_1^- & 0 \\ P_2^- & 1 \end{pmatrix}^{-1} \begin{pmatrix} y_1(0) - \overline{y}_1 \\ 0 \end{pmatrix} = \\ &= \frac{1}{P_1^-} \begin{pmatrix} 1 & 0 \\ -P_2^- & P_1^- \end{pmatrix} \begin{pmatrix} y_{1,0} - \overline{y}_1 \\ 0 \end{pmatrix} = \\ &= \begin{pmatrix} 1 \\ -P_2^- \end{pmatrix} \frac{(y_{1,0} - \overline{y}_1)}{P_1^-}. \end{aligned}$$

□

In this case the initial value for $y_2(0)$ is determined

$$y_2(0) = \overline{y}_2 + \frac{P_2^-}{P_1^-}(y_{1,0} - \overline{y}_1)$$

where $\frac{P_2^-}{P_1^-}$ is the slope of $\mathcal{E}^-$ which is co-incident with the stable manifold $\mathcal{W}^s$.

Sometimes if we assume we know the initial value for variable y_2, $y_2(0) = y_{2,0}$ the difference $y_{2,0} - \left(\overline{y}_2 + \frac{P_2^-}{P_1^-}(y_{1,0} - \overline{y}_1)\right)$ is interpreted as the initial "jump" to the saddle path.

4.6.4 Example: non-autonomous scalar equations

The ODE in subsection 4.5.2 , equation(4.38), can be a component of the of a specific initial-value problem we have considered in chapter two:

$$\begin{cases} \dot{y} &= \lambda\, y + \beta\, z \quad \text{for } \ t \geq 0 \\ \dot{z} &= \gamma\, z \quad \text{for } \ t \geq 0 \\ y(0) = y_0 & \text{for } \ t = 0 \\ z(0) = 1 & \text{for } \ t = 0. \end{cases} \tag{4.43}$$

Recall that we assume that $\lambda < 0$ and $\gamma > 0$. Using the general solution of the ODE in equation (4.38), we have the particular solution to this problem

$$\begin{aligned} y(t) &= \left(y_0 - \frac{\beta}{\gamma - \lambda}\right) e^{\lambda t} + \frac{\beta}{\gamma - \lambda}\, e^{\gamma t} \\ z(t) &= e^{\gamma t}. \end{aligned}$$

Then it is clear that although the phase diagram is a saddle, we are interested in a non-stationary solution

$$\lim_{t \to \infty} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \lim_{t \to \infty} \begin{pmatrix} \frac{\beta}{\gamma - \lambda}\, e^{\gamma t} \\ e^{\gamma t} \end{pmatrix} = \begin{pmatrix} \pm\infty \\ \pm\infty \end{pmatrix}$$

The solution converges to infinity to a line in the (z, y) space with slope $y = \frac{\beta}{\gamma - \lambda}\, z$. We are considering a **forward** ODE.

Alternatively, we could consider a **backward** ODE by specifying the problem

$$\begin{cases} \dot{y} &= \lambda\, y + \beta\, z \quad \text{for } \ t \geq 0 \\ \dot{z} &= \gamma\, z \quad \text{for } \ t \geq 0 \\ y(0) = y_0 & \text{for } \ t = 0 \\ \lim_{t \to \infty}\ e^{-\gamma t}\ z(t) = 0 & \text{for } \ t \to \infty. \end{cases}$$

In this case, the particular solution is

$$\begin{aligned} y(t) &= \ \left(y_0 - \frac{\beta}{\gamma - \lambda}\right) e^{\lambda t} \\ z(t) &= 0 \end{aligned}$$

which asymptotically converges to the steady state $\lim_{t \to \infty}\ (y(t), z(t)^{\top} = \mathbf{0}$.

4.7 Applications in Macroeconomics

Using the previous classification of ODE's we can offer a brief summary of applications in Economics

4.7.1 Pre-RE macroeconomics models

ISLM models which where the benchmark models in macroeconomics between early 1950's and middle 1970's (and still are the core of undergraduate macroeconomic courses) were static (non-dynamic) models. Introduction of dynamics in ISLM mocdels took the form of sluggish adjustment of some variables. A central aspect of those models, which make them the target of the Lucas critique, is that they assume what we can call static expectations: agents are now aware or have no beliefs concerning the future state of the economy, and, in particular on the consequences of economic policy.

These models have been called **ad-hoc macro models**. The reason is that they lack consistency when modelling agents' participation in several markets. This is not realistic, because, for instance, purchases in the goods markets should be financed, which means that product demand and money demand are linked in a particular way.

According to our previous definitions, ad-hoc dynamic models were usually initial-value problems in which the dynamic system is a stable node or stable focus (see see Takayama (1994), Turnovsky (1977), Gandolfo (1997) or Tu (1994)). The next example is a typical pre-RE macro model.

Example Assume aggregate private is $D(y, r) + g$, where g is government expenditure and the aggregate privte demand is a function of income and the real interest rate: $D(y, r) = d_0\, y - d_1\, r$, where $0 < d_0 < 1$ and d_1. Aggregate supply y is exogenous. There are two asset markets: a market for credit, a bond market, and a money market. By the Walras law we only need to model clearing of the money market. The demand for money is a function of income and the interest rate, $L(y, r) = l_0\, y - l_1\, r$, where $l_0 > 0$ and $l_1 > 0$, and the supply of money, m, is exogenous. It is assumed that both markets do not clear instantaneously, but price are constant, implying there is a temporary disequilibrium in both of them: $\dot{y} \ = \gamma_1(D(y, r) - y)$, and $\dot{r} \ = \gamma_2(L(y, r) - m)$. Therefore, we have a planar linear ODE, in which both variables are pre-determined,

$$
\begin{aligned}
\dot{y} \ &= \gamma_1 \Big(D(y, r) + g - y \Big) = \gamma_1 \Big(-(1 - d_0)\, y - d_1\, r + g \Big) \\
\dot{r} \ &= \gamma_2 \Big(L(y, r) - m \Big) = \gamma_2 \Big(l_0\, y - l_1\, r - m \Big) \\
y(0) &= y_0 \text{ given} \\
r(0) &= r_0 \text{ given.}
\end{aligned}
$$

Exercise Prove that there is a unique steady state and that it can be a stable node or focus. Consider the simplifying assumption $d_1 = l_0 = 0$. In this case prove that the steady state is a stable node. Furthermore, show that in this case the steady state levels satisfy $\overline{y} \ = \overline{y}(g)$ and $\overline{r} \ = \overline{r}(m)$. This means that the fiscal policy is efficient for controlling y and the monetary policy is only efficient for controlling r. This was in the center of the debate, which lasted for three decades, between the monetarists and keynesians.

4.7.2 Post RE ad-hoc macroeconomic models

In the early seventies it became clear, particularly because of the behavior of currency markets, when the Bretton Woods system came close to its end, that agents behavior depends on their beliefs. The simplest way to introduce beliefs is by assuming there could only be one right belief at the aggregate level, and that aggregate belief should be consistent with the our model of the economy. This is the origin of the designation rational expectations. The Dornbusch (1976) model became a benchmark, for RE ad-hoc macroeconomic models.

These models have again an ad-hoc structure where the dynamics is generated by the existence of slow adjustments for some variables and of perfect foresight for variables which translate beliefs. Mathematicaly, they are initial-terminal value problems in which the dynamic system is a saddle.

Example. The Dornbusch (1976) model formalizes the macroeconomic fluctuation in an open economy, in both the product and the asset markets, in which there are free movements of capital, and there is a flexible exchange rate regime. A simplified version of the model is the following.[12]

The nominal exchange rate (national currency per unit of foreign currency) is determined by the Fisher open equation in which its expected change is equal to the difference between the domestic and the foreign nominal interest rates $\dot{e} = E_t\left[\frac{de(t)}{dt}\right] = i - i^*$. The domestic nominal interest rate is determined in the equilibrium of the money market, which clears instantaneously. The real supply of money is $m - p = \ln M/P$, where m is the log of the nominal money supply and p is the log of prices, and the demand for real cash balances is Keynesian $L(i) = i$, to simplify. In the product market, the real aggregate demand is a function of income, of the nominal interest rate, and the real exchange rate (assuming that the log of the international price satisfies $p^* = 0$), $d(y, e, i, p) = \mu\,(e - p) - \sigma\, i + \delta\, y$, where $\mu > 0$, $\sigma > 0$ and $0 < \delta < 1$, and the real aggregate supply, y, is exogenous. The adjustment of the goods market is sluggish, but, differently from the previous model, this economy has flexible prices: $\dot{p} = \gamma(d(y, e, i, p) - y)$ if there is excess demand (supply) prices increase (decrease).

The following planar linear ODE, in which p is a pre-determined variable and e is non-predetermined variable is obtained

$$\begin{aligned}
\dot{p} &= \gamma\left(-(\mu+\sigma)\,p + \mu\, e + \sigma m - (1-\delta)\,y\right) \\
\dot{e} &= p - m - i^* \\
\pi(0) &= \pi_0, \text{ given} \\
\lim_{t\to\infty} e(t) &= \bar{e},
\end{aligned}$$

where $\bar{e}$ is the steady state level for the nominal exchange rate. In this case we say it is driven by the fundamentals.

[12] See Turnovsky (1995) for more RE models.

Exercise Prove that there is a unique steady state and that it is a saddle point. Furthermore, show that given an initial level for $p(0) = p_0$ there is only one trajectory consistent which this model, that is, there is one unique rational expectations path (draw the phase diagram). Assume that the economy is initially at a steady state (i.e., $p_0 = \bar{p}$), and assume there is an unanticipated, permanent and constant increase in the money supply m. Show that that the adjustment path involves "overshooting": there is an initial excess response of the nominal interest rate.

4.7.3 Optimizing economies or representative-agent DGE models

Ramsey (1928), and its rediscovery in the second half of the sixties by Cass (1965) and Koopmans (1965) started a strand of so-called non-ad-hoc modelling in macroeconomics. The inconsistency of the ad-hoc macro models has been solved by assuming that the economy is populated by homogeneous agents and it behaves efficiently. Although the difference between normative and positive economics is sometimes not clear in particular applications, these models are at the origin of what is called today representative-agent DGE (dynamic general equilibrium) models.

These models feature an initial-terminal value problem which is obtained from the first-order conditions of optimal control problems. When the equilibrium is Pareto optimal, the optimality conditions (or the equilibrium in a DGE interpretation) is represented by include both a forward belief (pre-determined) variable and a backward resource (non-predetermined) variable.

In these models the dynamic system is a saddle point or a saddle-node. The Ramsey problem is

$$
\begin{aligned}
\dot{k} &= F(k) - c \\
\dot{c} &= \frac{c}{\sigma}\left(r(k) - \rho\right) \\
k(0) &= k_0 \text{ given} \\
\lim_{t\to\infty} & \; k(t)\, c(t)^{-\sigma} \; e^{-\rho t} = 0
\end{aligned}
$$

where k is the stock of capital (a pre-determined variable) and c is consumption (a non-predetermined variable). The first ODE represents the budget constraint, in which capital accumulation is equal to savings. The second ODE is an Euler equation, or an intertemporal arbitrage condition. It takes the form of an arbitrage condition between present and future consumption, by equating the change in marginal utility by present consumption and the net increase in production capacity which will increase future consumption

$$\frac{du'(c(t))}{u'(c(t))} = \left(F'(k) - \rho\right) dt,$$

where ρ is the rate of time preference which measures impatience. The terminal constraint is called by economists the transversality condition and introduces a sustainability constraint on capital accumulation. In these models there is again an direct relationship between uniqueness of the saddle path, for a given initial level of the stock of capita, and the existence and uniqueness of an optimum (or DGE) path.

As this model is non-linear we need some results from non-linear ODEs to prove that the solutions os linear ODE provide a qualitative (although not a quantitative exact) solution to those models.

For references see Blanchard and Fischer (1989) and Turnovsky (1995).

4.7.4 Neo-Keynesian DGE models and non-representative agent DGE models

This structure allows for the both forward (pre-determined) and backward (non-predetermined or expected) dynamics, for the existence of DGE paths but non necessarily for their uniqueness. If DGE paths are not unique the dynamics is said to be indeterminate, meaning that self-fulfilling prophecies are possible, and these are related with the existence of imperfections in the markets (externalities, incompleteness of contracts, policy rules, etc).

Example A simple example which could be seen as an extension of the Ramsey model could be build by assuming that there are both externalities in production (positive or negative) and consumption (positive or negative), modelled by K and C. The DGE can be represented by

$$\begin{aligned} \dot{k} &= F(k, K) - c \\ \dot{c} &= \frac{c}{\sigma}\left(r(k, K) - \rho - \beta \frac{\dot{C}}{C}\right) \\ k(0) &= k_0 \text{ given} \\ \lim_{t\to\infty} & \; k(t)\, c(t)^{-\sigma} \; e^{-\rho t} = 0 \end{aligned}$$

the model is closed by a micro-macro consistency condition $K = k$ and $C = c$. We will see in one of the next chapters that this model may involve indeterminacy, that is the steady state can be a stable node or a stable focus for some values of the parameters.

4.7.5 Endogenous growth models

Endogenous growth theory models: are usually initial or initial-terminal value problems in which there are no positively valued steady states or steady states are a degenerate node (with a zero and a positive eigenvalue). Two-dimensional endogenous growth models usually feature dynamic systems with a zero and a positive real eigenvalue which is associated with the existence of a balanced-growth path.[13]

$$\begin{aligned} \dot{K} &= A\, K - C \\ \dot{C} &= C\,(A - \rho) \\ K(0) &= k_0 \\ \lim_{t\to\infty} & \; \frac{K(t)}{C(t)}\, e^{-\rho\, t} = 0 \end{aligned}$$

[13]See Acemoglu (2009)

Defining $K(t) = k(t)\, e^{\gamma t}$, $C(t) = c(t)\, e^{\gamma t}$ where $\gamma = A - \rho$ we obtain the problem in detrended variables (k, c) as an initial-terminal value problem

$$\begin{aligned} \dot{k} &= \rho\, k - c \\ \dot{c} &= 0 \\ k(0) &= k_0 \\ \lim_{t\to\infty} \frac{k(t)}{c(t)}\, e^{-\rho t} &= 0 \end{aligned}$$

As the solution of the detrended system is $k(t) = k_0 e^{\gamma t}$, and $c(t) = \rho\, k(t)$, therefore the solution of the AK model is

$$K(t) = k_0\, e^{\gamma t},\ C(t) = \rho\, k_0\, e^{\gamma t}$$

is called a balanced-growth path. Notice that the coefficient matrix of the detrended system is

$$\mathbf{A} = \begin{pmatrix} \rho & -1 \\ 0 & 0 \end{pmatrix}$$

has $\det(\mathbf{A}) = 0$ and $\text{trace}(\mathbf{A}) = \rho > 0$. This is a degenerate unstable-node. However, in this case this has welcome properties: degeneracy means that C is mononously related with K, and instability means that there is long run growth, i.e, the economy grows as a positive growth rate and becomes unbounded only in infinite time. This model is said to be a model of endogenous growth because the growth rate of the economy is not given in advance.

4.7.6 Bifurcation analysis and comparative dynamics

In economic applications we are interested in modelling the change in the trajectories of the state variables of interest when an exogenous variable or a parameter change. The need to study their variation can have different natures, although, mathematically, they are both parameters. in economic applications a parameter can be classified as an exogenous variables when it can be manipulated by a decision maker, while a parameter formalizes deep economic behaviors which can be determined with more or less precision. While varying the first allows the modeller have some insight regarding changes in policy, by varying the second we can have a measure on the robustness of our predictions.

We say we perform a **comparative dynamics** exercise when the variation of a parameter (in the mathematical sense) does not entail a change in the qualitative dynamics of the model, that is on the nature of its phase diagram.

This is different from **bifurcation analysis** in which we are interested in finding the : qualitative changes in the dynamics for variation of parameters (i.e, changes in the phase diagram for different values of the parameters)

4.8 References

Mathematical textbooks: Hirsch and Smale (1974), (Hale and Koçak, 1991, ch 8) and Perko (1996)

Economics textbooks: on dynamical systems applied to economics (Gandolfo (1997), Tu (1994)), general mathematical economics textbooks with chapters on dynamic systems (Simon and Blume, 1994, ch. 24,25), de la Fuente (2000).

4.A Appendix

4.A.1 Polar coordinates

When the eigenvalues are complex (or the model is non-linear) sometimes we can simplify the solution and get a better geometrical intuition of it, if we transform the ODE from cartesian coordinates $(y_1, y_2) \in \mathbb{R}$ into polar coordinates (r, θ) by using the transformation:

$$y_1(x) = r(x)\cos(\theta(x)),\ y_2(x) = r(x)\sin(\theta(x)).$$

where r measures the distance from a reference point (the radius) and θ the angular coordinate.

The following relationships hold $r^2 = y_1^2 + y_2^2$, because $\cos(\theta)^2 + \sin(\theta)^2 = 1$ and $\tan(\theta) = \sin(\theta)/\cos(\theta) = y_2/y_1$. If we take time derivatives of this two relationships we find

$$\begin{aligned} r' &= \frac{y_1\dot{y}_1 + y_2\dot{y}_2}{r} \\ \theta' &= \frac{y_1\dot{y}_2 - y_2\dot{y}_1}{r^2} \end{aligned}$$

Exercise: provide a proof (hint $d(\tan(\theta(x))/dt = (1 + \tan(\theta)^2)\theta' = (1 + (y_2/y_1)^2)\theta'$.

In order to apply this transformation, consider the ODE

$$\begin{aligned} \dot{y}_1 &= \alpha y_1 + \beta y_2 \\ \dot{y}_2 &= -\beta y_1 + \alpha y_2 \end{aligned}$$

The ODE in polar coordinates becomes

$$\begin{aligned} r' &= \alpha r \\ \theta' &= -\beta \end{aligned}$$

which has the general solution

$$\begin{aligned} r(x) &= r_0 e^{\alpha x} \\ \theta(x) &= \theta_0 - \beta x \end{aligned}$$

If $\alpha < 0$ the radius converges to zero (meaning that the the dynamics is stable) and if $\theta > 0$ the movement is clockwise.

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