EMA 2019-2020:
Problem set 2: linear planar ODE's

Paulo Brito
pbrito@iseg.ulisboa.pt

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1 Linear planar ODE's

1.1 General

2.1.1 Let $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where $\mathbf{A}$ can take one of the following values

$$\text{a)} \begin{pmatrix} -3 & 1 \\ -1 & -5 \end{pmatrix}, \text{ b)} \begin{pmatrix} -3 & 2 \\ -1 & -6 \end{pmatrix}, \text{ c)} \begin{pmatrix} -4 & 4 \\ -2 & -4 \end{pmatrix},$$

(a) Solve the planar ODE's and characterise analytically and geometrically the solutions for each case.

(b) Let $\mathbf{B} = (1, 1)$. Solve the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$, for the three cases, and characterise analytically and geometrically the solutions.

(c) Consider the ODE's of the last question. Let $\mathbf{y}(0) = (0, 0)$. Solve the initial value problems. Characterise analytically and geometrically the solutions of the initial value problem.

2.1.2 Let $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where $\mathbf{A}$ can take one of the following values

$$\text{a)} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \text{ b)} \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, \text{ c)} \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \text{d)} \begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix}, \text{e)} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix},$$

(a) Solve the planar ode and characterise analytically and geometrically the solutions

(b) Let $\mathbf{B} = (1, 1)$. Solve the planar ode $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$ and characterise analytically and geometrically the solutions

(c) Consider the ODE's of the last question. Let $\mathbf{y}(0) = (0, 0)$. Solve the initial value problems. Characterise analytically and geometrically the solutions of the initial value problem.

2.1.3 Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ a & -2 \end{pmatrix},$$

for $a \in \mathbb{R}$. Let a take any value on its domain. Determine the different solutions and characterise them.

2.1.4 Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} a & 1 \\ 1 & -3a \end{pmatrix},$$

for $a \in \mathbb{R}$. Let a take any value on its domain. Determine the different solutions and characterise them.

2.1.5 Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

(a) Solve the ODE.
(b) Draw the phase diagram and characterize it.
(c) Let $\mathbf{y}(0) = (0, 1)$. Solve the initial value problem.

2.1.6 Let $y = y(t)$ is a function, $y : \mathbb{R}_+ \to \mathbb{R}^2$. Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}.$$

(a) Solve the ODE.
(b) Draw the phase diagram and characterize it.
(c) Let $\mathbf{y}(0) = (-1, 1)$. Solve the initial value problem.

2.1.7 Consider the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} a & b \\ 0 & -b \end{pmatrix}.$$

where a and b are arbitrary constants.

(a) Which type of dynamics one would have.
(b) Let $a + b \neq 0$. Solve the ODE.
(c) Let a and b be strictly positive. Find, and characterize (as regards existence and uniqueness) the solutions converging asymptotically to $\mathbf{y} = (0, 0)^\top$.

2.1.8 Consider the two planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where $\mathbf{y} \in \mathbb{R}^2$

$$\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

and

$$\mathbf{A} = \begin{pmatrix} \lambda & \beta \\ \beta & \lambda \end{pmatrix}.$$

where λ and β are arbitrary constants.

(a) Solve the two ODE's, assuming that $\lambda \neq 0$.

(b) Perform a bifurcation analysis.

(c) Let $\lambda < 0$ and $\beta < 0$. Draw the phase diagrams and characterize them.

2.1.8 Consider the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$ where $\mathbf{y} \in \mathbb{R}^2$

$$\mathbf{A} = \begin{pmatrix} \alpha & 0 \\ 1 & \beta \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}$$

where α, β and γ are arbitrary real numbers.

(a) Study the possible effects on the dynamics of the solution for different values of those three parameters.

(b) Let $\beta < 0 < \alpha$ and $\gamma < 0$. Draw the phase diagram.

(c) Let $\mathbf{y}(0) = \mathbf{0}$. Find the solution of the initial value problem.

2.1.8 Consider the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$ where $\mathbf{y} \in \mathbb{R}^2$

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -\alpha \\ 0 \end{pmatrix}$$

where α and β are arbitrary real numbers.

(a) Study the possible effects on the dynamics of the solution for different values of those parameters.

(b) Let $\beta = 0$ and $\mathbf{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find the solution to the initial value problem.

2.1.8 Consider the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$ where $\mathbf{y} \in \mathbb{R}^2$

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(a) Solve the ODE.

(b) Draw the phase diagram. Present the analytical expressions for the isoclines and the eigenspaces.

(c) Assume that $y_2(0) = 0$ and that $\lim_{t\to\infty} y_1(t) = -\frac{1}{2}$. Find the particular solution.

2.1.8 Consider the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$ where $\mathbf{y} \in \mathbb{R}^2$

$$\mathbf{A} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(a) Find the steady state (if it exists) and characterize it.

(b) Draw the phase diagram. Justify it analytically.

(c) Assume that we have the initial value $\mathbf{y}(0) = (y_1(0), y_2(0))^\top = (0, 1)^\top$. Find the solution to the Cauchy problem.

2.1.8 Consider the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$ where $\mathbf{y} \in \mathbb{R}^2$

$$\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

(a) Solve the ODE.

(b) Draw the phase diagram. Justify it analytically.

(c) Assume the initial value $y_1(0) = 2$. Find the solution(s) converging to the steady state.

2.1.8 Consider the planar ODE for $\mathbf{y} : \mathbb{R}_+ \to \mathbb{R}^2$, with the independent variable $t \in \mathbb{R}_+$,

$$\begin{aligned} \dot{y}_1 &= -y_1 + 2\, y_2 + \alpha, \\ \dot{y}_2 &= y_1 - 2\, y_2. \end{aligned} \tag{1}$$

(a) Assume that $\alpha = 0$. Draw and characterize the phase diagram.

(b) Assume that $\alpha = 0$. Solve the equation analytically.

(c) Assume that $\alpha = 1$. Draw the phase diagram and characterize it.

(d) Prove that the solution to the ODE of the equation (1) verifies $y_1(t)+y_2(t)-\alpha\, t = y_1(0) + y_2(0)$ for any $t \in \mathbb{R}_+$.

1.2 Applications

2.2.1 Consider a continuous time version of a two-state Markov process $\dot{y} = My$, where the transition matrix is

$$M = \begin{pmatrix} p-1 & 1-p \\ 1-q & q-1 \end{pmatrix}$$

for $0 < p < 1$ and $0 < q < 1$

(a) solve the differential equation;

(b) let $y(0) = (1, 2)$. Solve the initial value problem;

(c) draw the phase diagram.

2.2.2 Consider a continuous time version of a two-state Markov process $\dot{y} = My$, where the transition matrix is

$$M = \begin{pmatrix} -\pi_1 & \pi_1 \\ \pi_2 & -\pi_2 \end{pmatrix}$$

for $0 < \pi_1 < 1$ and $0 < \pi_2 < 1$

(a) solve the differential equation;

(b) let $y(0) = (0, 1)$ and solve the initial value problem;

(c) draw the phase diagram associated to the initial value problem.

2.2.3 Consider the wage-price dynamics for an economy in which there is perfect foresight in the product market and in which wages do not fully adjust to imbalances in the labour market. The economy is represented by a planar ODE

$$\begin{aligned}\dot{p} &= \lambda(p - m)\\ \dot{w} &= \gamma(p - w - N)\end{aligned}$$

where λ and γ are positive parameters and m and N are exogenous variables (money and population respectively). Assume that $w(0) = w_0$ is given and that prices verify $\lim_{t\to\infty} p(t) = \overline{p}$:

(a) determine the fixed point;

(b) solve the ODE;

(c) solve the mixed initial-terminal value problem;

(d) draw the phase diagram;

(e) which consequences will arise from an increase in the money supply ?

2.2.4 This is inspired in the Calvo (1983) model. Assume an imperfectly competitive economy in which the firms have the following rule for setting prices: $x(t) = \delta \int_t^{+\infty} (p^*(s) + \beta y^*(s)) e^{-\delta(t-s)} ds$, for $\beta > 0$ and $\delta > 0$, where x is the price set by each firm, p is the aggregate price index, y is the aggregate level of activity, and δ denotes the (constant) probability for price revisions. All the variables are logarithms and the star represents expectations. Differentiating, we have equivalently $\dot{x} = \delta(x - p^* - \beta y^*)$. We assume that the aggregate price level is a weighted average of the prices set by all firms and it is given by $p(t) = \delta \int_{-\infty}^{t} x(s) e^{-\delta(t-s)} ds$, or equivalently $\dot{p} = \delta(x - p)$.

Equilibria in the goods and monetary markets implies that the following reduced form equation holds $y(t) = a(m(t) - p(t)) + b\pi^*(t)$, where m is the (log) of the money stock and $\pi \equiv \dot{p}$ is the inflation rate. The nominal money stock m is constant and exogenous. At last, assume that in this economy agents have perfect foresight (i.e, $p^* = p$). All the parameters are positive.

a) Obtain a planar ODE over (p, x), representing this economy

b) Perform a qualitative analysis of the local dynamics. Assume that $\beta b < 1 < \beta(a + b)$.

c) Assume there is a non-anticipated and permanent shock in m. Study the comparative dynamics assuming that x is non-predetermined and p is pre-determined.

d) Discuss the goodness of the choice of x as a non-predetermined variable, versus the alternative in which p is non-predetermined. Does it makes sense to assume that both variables are non-predetermined ? What would be the comparative dynamics in this case ?

2.2.5 The Dornbusch (1976) was, possibly, the most popular macroeconomic model between the second half of the seventies and most part of the eighties (at least). It is representative of the "rational expectations revolution" before the DGSE models became the benchmark model in dynamic macroeconomics. It is is a model for an open economy with following features: (1) there is free international movements of capital and the domestic interest rate i adjusts to (exogenous) international interest rate i^* through an open-Fisher equation; (2) there is rational expectations concerning the rate of depreciation, $x = \dot{e}$, where x is the expected depreciation rate and $\dot{e}$ is the market rate of depreciation; (3) the only policy instrument is the aggregate money supply m; (4) the aggregate supply is given ($\bar{y}$) but the prices (p) adjust sluggishly to the excess demand in the product market. The equations of the model assume typically Keynesian behavioral assumptions:

$$\begin{aligned}
i &= i^* + x \\
x &= \dot{e} \\
m - p &= -\xi i + \phi y \\
d &= g + \delta(e + p^* - p) + \gamma y - \sigma i, \ 0 < \gamma < 1 \\
y &= \bar{y} \\
\dot{p} &= \pi(d - y),
\end{aligned}$$

all the parameters are positive and all the variables are in logs.

(a) Prove that those equations reduce to a linear planar ODE

$$\begin{aligned}
\dot{e} &= \frac{1}{\xi}\Big(p - m + \phi y\Big) - i^*, \\
\dot{p} &= \pi\left[\delta e - \Big(\delta + \frac{\sigma}{\xi}\Big) p + \frac{\sigma}{\xi} m + \Big(\gamma - 1 - \frac{\phi\sigma}{\xi}\Big) y + \delta p^* + g\right].
\end{aligned}$$

(b) Find the steady state and study the local dynamics

(c) Draw the phase diagram and explain.

(d) Solve the model assuming that $p(0) = p_0$ is given and $\lim_{t\to\infty} e(t) = \bar{e}$.

(e) Perform a dynamic comparative analysis of a permanent, non-anticipated, positive shock in m.

2.2.5 The AK model (see Jones and Manuelli (2005) for a survey) is at the same time the simplest and the benchmark endogenous growth model. The endogenous variables are consumption C and the stock of capital K and the optimal conditions are a planar ordinary differential equation

$$\begin{aligned}
\dot{C} &= C\Big(\frac{A - \delta - \rho}{\theta}\Big) \\
\dot{K} &= A\,K - C - \delta\,K
\end{aligned}$$

together with the initial conditions

$$K(0) = K_0 \text{ and } \lim_{t\to\infty} \frac{K(t)}{C(t)^{\theta}} e^{-\rho t} = 0$$

where $A > 0$ is the total factor productivity, $\delta > 0$ is the depreciation rate and $\rho > 0$ is the rate of time preference. We assume that state space is $\mathbb{R}^2_{++}$.

(a) Study the existence and uniqueness of steady states and its (their) local dynamics properties.

(b) Draw the phase diagram.

(c) Find the explicit solution to the planar ODE.

(d) Find the explicit solution of the problem.

(e) The aggregate income be $Y(t) = AK(t)$. Discuss the characteristics of the growth process generated by this model.

References

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