AME 2022-2023:
Problem set 6: Stochastic differential equations

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1 General

1. The diffusion equation is

$$dX(t) = \gamma X(t)dt + \sigma X(t)dW(t)$$

for $X(0) = x_0$.

(a) Prove that the solution is $X(t) = x_0 e^{(\gamma - \sigma^2/2)t + \sigma W(t)}$

(b) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

(c) Derive the backward Kolmogorov equation for the probability for $X(T) \leq 2x$ assuming that $X(t) = x$

(d) Derive the forward Kolmogorov equation for the density associated to $X(t) = x > 0$, assuming that $X(0) = 0$.

2. Consider the diffusion equation

$$dX(t) = \gamma X(t)dt + \sigma dW(t)$$

where $\{W(t)\}$ is a standard Brownian motion.

(a) Let $X(0) = x_0$ be known. Find the solution of the initial value problem.

(b) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

(c) Write the forward Kolmogorov equation for the density associated to $X(t) = x$. Provide an intuition for this equation.

3. Consider the diffusion equation

$$dX(t) = \big(\sigma_0 + \sigma_1 X(t)\big)\, dW(t)$$

where $\{W(t)\}$ is a standard Brownian motion.

(a) Let $X(0) = x_0$ be known. Find the solution of the initial value problem.

(b) Write the forward Kolmogorov equation for the density associated to $X(t) = x$. Solve it.

(c) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

4. Consider the problem

$$dX(t) = \mu\, dt + \sigma\, \sqrt{X(t)}\, dW(t),\ t > 0$$
$$X(0) = x_0$$

where $\{W(t)\}$ is a standard Brownian motion, and μ and σ are constants.

(a) Find the process for $Y(t) = \sqrt{X(t)}$.

(b) Write the Fokker-Planck-Kolmogorov for $\mathbb{P}[X(t) = x|X(0) = x_0]$. Solve it.

(c) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

5. Consider the diffusion equation

$$dX(t) = (1 - X(t))dt + dW(t),\ t \in [0, \infty)$$

where $\{W(t)\}$ is a standard Wiener process.

(a) Let $X(0) = 0$. Find the solution of the initial value problem.

(b) Find $\mathbb{E}[X(t)|X(0) = 0]$ and $\mathbb{V}[X(t)|X(0) = 0]$.

(c) Write the forward Kolmogorov equation for the density associated to $X(t) = x$. Provide an intuition for this equation.

6. Consider the diffusion equation

$$dX(t) = -X(t)dt + dW(t),\ t \in [0, \infty)$$

where $\{W(t)\}$ is a standard Wiener process.

(a) Let $X(0) = x_0$, where x_0 is a real number. Find the solution to the initial value problem.

(b) Write the forward Fokker-Planck-Kolmogorov (FPK) for the density associated to $X(t) = x$, conditional on $X(0) = x_0$, that is $p(t, x) = \mathbb{P}[X(t) = x\,|X(0) = x_0]$.

(c) Let $P(t, s) = \mathcal{F}[p(t, x)]$ be the Fourier transform of $p(t, w)$. Find $P(t, s)$, which is the solution to the transformed FPK equation together with the initial condition $P(0, s) = \mathcal{F}[\delta(x - x_0)] = e^{-2\,\pi\, i\, s\, x_0}$(tip: $\mathcal{F}[x\,\partial_x p(t, x)] = -\big(P(t, s) + s\,\partial_s P(t, s)\big)$ and $\mathcal{F}[\partial_{xx} p(t, x)] = -(2\,\pi\, s)^2\, P(t, s)$).

7. Consider the stochastic differential equation

$$dX(t) = (a - X(t))dt + b\, dW(t),\ t \in [0, \infty)$$

where $\Big(W(t)\Big)_{t\geq 0}$ is a standard Wiener process, and a and b are positive constants.

(a) Let $X(0) = a$. Find the solution of the initial value problem (hint: use the transformation $Y(t) = X(t)\, e^t$).

(b) Write the Fokker-Planck-Kolmogorov equation for $\mathbb{P}[X(t) = x|X(0) = a]$. Solve it.

(c) Find $\mathbb{E}[X(t)|X(0) = a]$ and $\mathbb{V}[X(t)|X(0) = a]$. Provide an intuition for their asymptotic values (i.e., for their limit when $t \to \infty$).

8. Consider the following problem

$$\begin{aligned} dX(t) &= \left(\frac{\sigma}{2}\right)^2 dt + \sigma\, \sqrt{X(t)}\, dW(t),\ t \in (0,\, T] \\ X(T) &= a^2, \end{aligned}$$

where $\{W(t)\}$ is a standard Brownian motion, and σ, a and T are positive numbers. In order to solve the problem, consider the following steps:

(a) Find the process for $Y(t) = \sqrt{X(t)}$ by using Itô's lemma.

(b) Find the general solution for $X(t)$.

(c) Find the particular solution to the proposed problem.

2 Economic applications

1. The Vasicek 1977 (or Ornstein-Uhlenbeck) process is the solution of the equation

$$dX(t) = \theta(\mu - X(t))dt + \sigma dW(t)$$

for $X(0) = x_0$.

(a) Prove that the solution is

$$X(t) = \mu + (x_0 - \mu)e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s)} dW(s)$$

(b) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

2. Assume an AK model where $Y = A(t)K(t)$ where the productivity follows a SDE

$$dA(t) = \gamma dt + \sigma dW(t)$$

and the equilibrium equation is $dK(t) = sY(t)dt$, and $K(0) = k_0$ given. All the parameters, γ, σ and s are positive.

(a) Find the solution to the capital process $(K(t))_{t \in \mathbb{R}_+}$.

(b) Write the forward Fokker-Planck-Kolmogorov (FPK) for the density associated to $K(t) = k$, conditional on $K(0) = k_0 > 0$, that is $p(t, k) = \mathbb{P}[K(t) = k\,|K(0) = k_0]$.

(c) By solving the FPK equation find the conditional mean and variance of $K(t)$.

3. Consider a stochastic Solow model in which the population is constant and the production function is $Y(t) = A(t)K(t)^{\alpha}L^{1-\alpha}$ with $=0 < \alpha < 1$. The total factor productivity follows the process

$$dA(t) = \gamma A(t)dt + \sigma A(t)dW(t).$$

where $\gamma > 0$ and $\sigma > 0$ and $(W(t))$ is a Wiener process. The equilibrium in the product market is given by equation

$$\frac{dK(t)}{dt} = sY(t) - \delta K(t)$$

where $0 < s < 1$ is the savings rate and δ is the depreciation rate.

(a) Write the Itô stochastic differential equation for capital in intensive terms $k(t) \equiv K(t)/L(t)$

(b) Assuming that $K(0) = k_0$ and defining the conditional probability as $p(t,k) = \mathbb{P}[K(t) = k|K(0) = k_0]$ write the Fokker-Planck-Kolmogorov equation.

(c) Assume that the SDE you have found in (a) has the form $dk = \mu(k)dt + \sigma(k)dW(t)$. Consider an approximation of this SDE by the SDE $dk = \lambda\left(\bar{k} - k\right)dt + \sigma(k)dW(t)$, where $\bar{k}$ is the steady state of the squeleton (i.e., the value of $k > 0$ such that $\mu(k) = 0$ and $\lambda = \frac{\partial \mu}{\partial k}(\bar{k})$ (the derivative evaluated at that steady state). Find the Fokker-Planck-Kolmogorov equation.

(d) Find the asymptotic moments of the process of the approximated process and provide an intuition for your results.

4. Consider a stochastic Solow model in which the population is constant and the production function is $Y(t) = A(t)K(t)^{\alpha}L^{1-\alpha}$ with $=0 < \alpha < 1$. The total factor productivity follows the process

$$dA(t) = \gamma A(t)dt + \sigma A(t)dW(t).$$

where $\gamma > 0$ and $\sigma > 0$ and $(W(t))$ is a Wiener process. The equilibrium in the product market is given by equation

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(a) Write the Itô stochastic differential equation for capital in intensive terms $k(t) \equiv K(t)/L(t)$

(b) Assuming that $K(0) = k_0$ and defining the conditional probability as $p(t,k) = \mathbb{P}[K(t) = k|K(0) = k_0]$ write the Fokker-Planck-Kolmogorov equation.

(c) Assume that the SDE you have found in (a) has the form $dk = \mu(k)dt + \sigma(k)dW(t)$. Consider an approximation of this SDE by the SDE $dk = \lambda(\bar{k} - k)\,dt + \sigma(k)dW(t)$, where $\bar{k}$ is the steady state of the squeleton (i.e., the value of $k > 0$ such that $\mu(k) = 0$ and $\lambda = \frac{\partial \mu}{\partial k}(\bar{k})$ (the derivative evaluated at that steady state). Find the Fokker-Planck-Kolmogorov equation.

(d) Find the asymptotic moments of the process of the approximated process and provide an intuition for your results.

References

Vasicek, O. A. (1977). "An equilibrium characterization of the term structure". In: *Journal of Financial Economics* 5, pp. 177–88.