AME 2022-2023:
Problem set 7: Optimal control of stochastic differential equations

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1. For a given initial level of the capital stock $K(0) = k_0$, the stock of capital, $K(t)$, of a firm evolves according to the diffusion process

$$dK(t) = \big(I(t) - \delta\, K(t)\big)dt + \sigma\, dW(t),\ t > 0$$

where $I(t)$ is gross investment, $\delta > 0$ is the capital depreciation rate, and $\sigma\, dW(t)$ represents additive random shocks to the capital accumulation process, where $\{W(t)\}_{t\in\mathbb{R}_+}$ is a Wiener process.

The firm's objective is to use gross investment to maximize the expected value of the present value of the cash flows,

$$\mathbb{E}_0\left[\int_0^\infty \Big(AK(t) - \frac{I(t)^2}{2}\Big)e^{-rt}dt\right].$$

   (a) Write the Hamilton-Jacobi-Bellman equation.
   (b) Solve the HJB equation (hint: use a linear trial value function).
   (c) Obtain the stochastic differential equation (SDE) for the optimal capital accumulation equation.
   (d) Solve that equation. **Remark**: if you could not obtain the SDE in (c), use instead $dK(t) = \big(\bar{I} - \delta\, K(t)\big)dt + \sigma\, dW(t)$ where $\bar{I} > 0$ is a constant, in this question and in the following questions.
   (e) Find the statistics for the optimal capital stock of the firm (expected value and variance).
   (f) Provide an intuitive discussion of your results (hint: compare them with the deterministic analog).
   (g) Write the Fokker-Planck-Kolmogorov equation associated to the distribution of the capital stock, starting from $K(0) = k_0$, which is fully observed.

(h) Solve this equation for the case in which $\delta = 0$.

2. Consider the following stochastic resource-depletion problem, where $\{X(t)\}_{t\in\mathbb{R}}$ is the process for the stock of the resource, and $\{C(t)\}_{t\in\mathbb{R}}$ is the process for its use,

$$\max_{C(\cdot)} \mathbb{E}_0\Big[\int_0^\infty \ln\left(C(t)\right) e^{-\rho\, t}\Big]$$

subject to

$$dX(t) = -C(t)\, dt + \sigma\, X(t)\, dW(t), \text{ for } t \in (0, \infty)$$

where $\{W(t)\}_{t\in\mathbb{R}}$ is a Wiener process, and $X(0) = x_0 > 0$ is given. The rate of time preference and the volatility parameters, ρ and σ, are both positive and satisfy $\rho > \sigma^2$.

(a) Write the first-order conditions for optimality according to the stochastic Pontriyagin's maximum principle.

(b) Find the stochastic process for $C(t)$.

(c) Using $C(t) = \phi\, X(t)$, for an undetermined constant ϕ , as a trial function find the solution for the optimal $X(t)$.

3. Consider the diffusion equation

$$dX(t) = \big(u - \sigma\, X(t)\big)\, dW(t),\ X \in \mathbb{R},\ t \in \mathbb{R}_+$$

where u and $\sigma > 0$ are known constants and $\{W(t)\}$ is a standard Brownian motion. Let $X(0) = x_0$ be a known constant.

(a) Is there an absorbing state ? If this is the case denote it by $\bar{x}$. Find the solution to the initial value problem (Hint: define $Y(t) = X(t) - \bar{x}$ and solve first the transformed problem for $Y(t)$).

(b) Write the Fokker-Planck (or forward Kolmogorov) equation for the density associated to $X(t) = x$ (or $Y(t) = y$). Solve it. Find $\mathbb{E}_0[X(t)]$ and $\mathbb{V}_0[X(t)]$.

(c) Assume we want to control the volatility of the process $\{X\}$ by solving the stochastic optimal control problem

$$\max_{U(\cdot)} \mathbb{E}_0\Big[\int_0^\infty -U(t)^2\, e^{-\rho t}\, dt\Big],\ \rho > 0$$

$$\text{subject to}$$
$$dX(t) = \big(U(t) - \sigma X(t)\big)\, dW(t),\ t \in \mathbb{R}_+$$
$$X(0) = x_0 \text{ given.}$$

Find the optimal processes $\{U^*\}$ and $\{X^*\}$ (Hint: if you use the dynamic programming principle try the value function $v(x) = \beta\, x^2$). Find $\mathbb{E}_0[X^*(t)]$ and $\mathbb{V}_0[X^*(t)]$. Compare with the non-optimal statistics in (b).

4. Consider the problem

$$\max_{C,\theta} \mathbb{E}_0\Big[\int_0^\infty \ln\left(C(t)\right) e^{-\rho\, t} dt\Big],\ \rho > 0$$

subject to

$$dN = \Big(\big(r\,(1-\theta) + r^s\,\theta\big)\,N + y - C\Big)dt + \sigma\,\theta\,dW(t)$$

where we consider the process for consumption, C, net financial wealth, N, the portfolio composition, θ, and the Wiener process. The initial wealth $N(0) = n_0$ is given and wealth is asymptotically bounded. Solve the stochastic problem for a representative consumer assuming a log utility function, for the case in which the non-financial income is constant.

(a) Using the principle of dynamic programming.

(b) Using the stochastic Pontriyagin maximum principle.

5. Consider the following stochastic optimal control problem for a small open economy

$$\max_{C} \mathbb{E}_0\Big[\int_0^\infty \ln\left(c(t)\right) e^{-\rho\, t}\, dt\Big],\ \text{where}\,\rho > 0$$

$$\begin{aligned}&\text{subject to}\\ &dA(t) = \big(r\,A - \pi\,C\big)\,dt + \sigma\,A\,dW(t)\\ &A(0) = a_0 > 0, \text{given}\end{aligned}$$

where A is the net international asset position of the economy, C is total consumption (of domestically produced and imported goods), r and σ are the (constant) mean rate of return and volatility of international assets, and π is the (constant) relative price of consumption, and $\{W(t)\}$ is a Wiener process.

(a) Find the optimal policy function $C^* = C(A)$, by using the principle of dynamic programming.

(b) The budget constraint in the optimum is a stochastic differential equation. Find it and solve it.

(c) Let $p(t,a) = \mathbb{P}[A(t) = a|A(0) = a_0]$. Write the FPK equation for the distribution of A.

(d) Prove that

$$p(t,a) = \frac{a_0}{a\sqrt{2\pi\sigma^2 t}}\exp\Big\{-\frac{\big(\ln\left(a/a_0\right) + \frac{\sigma^2}{2} + \rho - r\big)^2}{2\sigma^2 t}\Big\}$$

(e) Find $\mathbb{E}_0[A(t)]$ and $\mathbb{V}_0[A(t)]$. Discuss your results.