The dynamics of growth and distribution in a spatially heterogeneous world

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Abstract

Distributional extensions of the benchmark AK endogenous growth model and of the Ramsey model are presented in this paper. The resulting geographic growth model - a forward-backward parabolic partial differential equation (PDE) over a bounded spatial domain - is governed by two main driving forces: a spatial friction in the reallocation of physical capital, and a spatial arbitrage driving the reallocation of savings.

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The spatial AK model solution, starting from an unequal distribution of capital, displays convergence over time to a spatially homogeneous balanced growth path with positive growth rates. The spatial Ramsey model potentially contains both diffusive and agglomerative spatial forces. If there are no agglomerative forces, the solution displays convergence to a spatially homogenous steady state, but if an agglomerative force exists, there is increasing spatial concentration of capital over time. This shows that convergence in the aggregate can be consistent with different distribution profiles over locations. An additional contribution of the paper is to develop a distributive comparative dynamics analysis for a spatially heterogeneous productivity shock. It is shown that, although there is redistribution of consumption across locations, technological inequality is persistent.

JEL Classification: C6, D9, E1, R1.

Keywords: distributional endogenous growth, spatial growth, optimal control of parabolic PDE, Fourier transforms, spectral bifurcation analysis.

1 Introduction

I try to answer the following questions:[1] how can geography be introduced in the forward-backward dynamics of the benchmark aggregative endogenous growth model ? Does spatial heterogeneity generate new types of endogenous growth dynamics ? More generally, how does distributional dynamics differ from aggregative dynamics ?

The benchmark aggregative, or a-spatial, AK-Ramsey model of economic growth assumes an autarkic environment in which there is a single location or an ensemble of isolated locations. The core of that model combines, in a consistent way, the accumulation of physical capital, a pre-determined variable, and an arbitrage condition for its marginal value, a forward-looking variable. In continuous-time, the first component is modelled by a forward ordinary differential equation (ODE), and the second component is modelled by a backward ODE. The forward dynamics is driven by the equilibrium between demand and supply of physical capital, where the demand of capital reduces to net investment and the supply of capital comes from savings. The backward dynamics is governed by the arbitrage condition between the marginal benefit and the marginal cost of investing in capital. While the marginal benefit is equal to the sum of the change in the marginal value plus the net rate of return of capital, the marginal cost is equal to the rate of time preference.[2] The wedge between the rate of return of capital and the rate of time preference, together with the pre-determined income from production, determines the change in savings, which feeds back into the forward dynamics of capital accumulation over time. The asymptotic dynamics of capital accumulation depends on the type of production technology. If there are constant returns to scale, in the AK model, the rate of return of capital is independent from the stock of capital. If the marginal productivity of capital is greater than the rate of time preference, there will be a permanent value wedge, which causes permanent positive savings leading to long-run growth. However, if there are decreasing returns to scale, in the Ramsey model,

[1]This paper is an updated version of Brito (2004) and Brito (2011).

[2]The backward dynamic is governed by the Hamilton-Jacobi-Bellman equation together with a transversality condition. Although it models a forward-looking behavior, it is mathematically a backward dynamic system.

capital accumulation reduces the rate of return of capital. This implies that the value wedge decreases over time which progressively eliminates the incentives to saving and cause the convergence of the economy towards a steady state.

In this paper I introduce a distributional AK-Ramsey model, in which there is an open economy environment and there are flows of physical and financial capital across locations.[3] Let us take the perspective from an arbitrary location within that environment. In addition to the previous forward-backward dynamics of the a-spatial or autarkic economy, two new spatial-related dimensions of the model should be introduced: a spatial reallocation of physical capital, and a spatially-related incentive for the reallocating savings across locations. In this model, the total demand for capital comes not only from the home location but also from all the other locations, which implies that the spatial reallocation of physical capital can either increase or reduce the supply of capital to a particular location. This implies that equilibrium between demand and supply of physical capital is now formalized by a forward, in time, partial differential equation (PDE). Furthermore, a new spatial wedge must be added to the time wedge between the marginal net productivity of capital and the rate of time preference, which is the only present in an autarkic economy. This wedge introduces a spatial arbitrage, when there is heterogeneity in the distribution of the marginal value of capital across locations, and can either increase or decrease the marginal value of the financial asset position for a given location. This implies that the arbitrage between marginal value and cost of capital is now formalized by a backward, in time, PDE. In an open economy, the dynamics of the marginal value of capital influences the spatial distribution of savings, and therefore the accumulation of capital over time and across locations.

In our model, the two forward-backward dynamic distributional elements are specified by two assumptions. First, I postulate a spatial friction in the transportation of physical capital, taking the form of a spatial adjustment cost, and this adjustment cost is proportional to the local curvature of the ensemble distribution of capital across all locations. In particular, if this

[3] It can be re-interpreted as referring to total capital, human and physical, if both types of capital are perfect substitutes. In this case, in the Ramsey version of the model the existence of a fixed resource should be assumed.

curvature is locally convex (concave) then there will be an inward (outward) reallocation of physical capital. Second, I assume that there are no externalities or other market distortions, which makes the economy, as in the AK-Ramsey a-spatial model, Pareto efficient, in the sense that intertemporal and inter-spatial allocation of consumption maximizes an intertemporal social welfare functional. To aggregate the intertemporal utility functionals of different locations, I assume a Millian, or uniformly weighted, welfare criterium. The associated Euler equation features consumption smoothing over time and across locations thereby avoiding the un-ethical spatially biased welfare aggregation, which is present in spatially discounted social welfare functionals.

Two versions of the model are presented, a distributed AK model and a distributional Ramsey model. The technology of production is the distinguishing assumption: there are constant returns to scale in the first case, and decreasing returns to scale in the second case. In the distributional AK model a diffusive dynamics is dominant causing the convergence of the ensemble distributions of capital over time to a spatially-homogeneous unbounded balanced growth path. In the distributional Ramsey model there is a tension between agglomerative and diffusive forces that, although originating different profiles of the distributions over time, allow for the convergence to a stationary bounded asymptotic distribution. Interestingly, in the last case, I show that, although we can have stability in the aggregate (in the L^1 sense), we can have two possible distinct distributional profiles, a stable one in which diffusion is dominant or an unstable one in which agglomeration prevails.

The distributional growth models in this paper are, mathematically, optimal distributed control problem constrained by a parabolic PDE. Consequently, the Hamiltonian dynamics is represented by a coupled forward-backward system of parabolic PDEs. I assume, in most of the paper, a finite-dimensional ring as the spatial domain. This allows for an analytical solutions, in closed form for the AK version and by linearization in the Ramsey version, by using finite Fourier transforms, from the set of locations to a set of frequencies. This method allows for a distributional bifurcation analysis of the time evolution and distributional features of the solutions in both models. In particular, I show that the overall stability is related to a frequency-independent growth factor and the existence of agglomeration or diffusion depends on the existence of spectral instability (in the sense of the existence of positive eigenvalues for some frequencies) or spectral stability (negative eigenvalues for the whole spectrum).

This model contains a very rich set of both intertemporal (forward and backward) and inter-spatial (transport- or arbitrage-related) components. One contribution of this paper is to show that that approach allows to a relatively simple determination of which components are dominant in driving the distributional dynamics. I find that the asymptotic boundedness of the distributions is determined by the classic relationship between the rates of return of capital and the rate of time preference, and the spectral stability is related to the relative strength of the spatial flow of both physical and financial capital and the intertemporal substitution in consumption. High barriers to the spatial flow of capital and a low elasticity of intertemporal substitution in consumption tend to lock local capital levels or productivity gains and generate agglomerative dynamics.

Another contribution of the paper is to apply this approach to present a comparative dynamics analysis of a spatially heterogeneous productivity shock for the distributional Ramsey model. Under the assumption of spectral stability, I show that, although the diffusive dimension prevails over time, there is convergence to a bounded asymptotic distribution in which both inequality in income distribution and some spatial redistribution of consumption availability exist. This implies the existence of both local heterogeneity in savings and zero savings at the aggregate level, that is of an asymptotic uneven distribution of current accounts across locations.

The integration of growth and geography is a conceptually difficult endeavour. It has been addressed recently by using PDEs. The idea of modelling geographic growth by using parabolic PDEs can be traced back to Isard and Liossatos (1979). In Brito (2004) and Brito (2011) I developed this idea further by using relatively simple classical PDE methods. By now, this approach became an increasingly populated strand of the literature, dealing rigorously with well-posedness, existence, uniqueness and the analytical properties of the solution.

In particular, Boucekkine et al. (2009), Ballestra (2016), Fabbri (2016), Boucekkine et al. (2013), Boucekkine et al. (2019a) present several versions of the model with an AK technology, with different domains for the spatial variable, different TFP technologies, different utility functionals, and different methods of proof. Versions of this model with decreasing returns to scale have been studied in Brito (2004), Brock and Xepapadeas (2008) , Brito (2011) , and Xepapadeas et al. (2020). The existence of a Turing (1952) bifurcation, usually associated to the emergence of pattern formation, has already been identified in Brito (2004) and Brock and Xepapadeas (2008). Also, in this strand of the literature, Fredrick et al. (2019) presents an extension incorporating non-local effects. Augeraud-Véron et al. (2019) surveys applications to resource economics.

The contribution of this model to spatial economics has already been surveyed in Desmet and Rossi-Hansberg (2010) (see also the survey in Breinlich et al. (2014)). Comparing to benchmark references in spatial economics, as Fujita et al. (1999), Fujita and Thisse (2002) and Fujita and Mori (2005), our model is distinguished by two main features: it deals with economic growth, not with spatial agglomeration of economic activity, and it incorporates the forward-looking behavior of agents, which is absent in those contributions. Surprisingly, the spatial dynamic modelling in this literature leads to an ODE in space, away from Beckmann (1970) and (Beckmann and Puu, 1985, ch. 1) who already modelled space-time dynamics by first-order PDEs.

Mean-field optimal control and mean-field games, following Lasry and Lions (2007) and surveyed by v.g., Gomes and Saúde (2014), have built an impressive body of literature using forward-backward PDEs (see Carmona and Delarue (2018) for a thorough presentation of methods and models). In economics, this approach provides an analytically tractable way for dealing with continuous-time heterogeneous-agents general equilibrium. Achdou et al. (2014) survey the contribution of this approach to economics, Nuño and Moll (2018) and Achdou et al. (2021) present applications to macroeconomics, and Brito (2019) to endogenous growth theory. Beyond the forward-backward PDE mathematical structure, the models in this literature share one important feature with this paper, but also differ in a crucial dimension. Similarly to our model, they feature a variation-type heterogeneity, not a variety-type heterogeneity, which means that the index-space has an order or vector-field structure. A vector-space structure of the index-space allows for modelling the distributional dynamics by PDEs. Variation-type heterogeneity can also be found in static models, as in assignment models, (as in Costinot and Vogel (2010)), adverse selection contract models (as in Salanié (2005)), occupational choice models (as in Lucas (1978)), or optimal taxation models (as in Mirrlees (1971)). But, differently from the model in this paper, mean-field related models combine a Fokker-Planck, or forward Kolmogorov, equation and a Hamilton-Jacobi-Bellman equation. The FPK equation provides the dynamics of a density function for a process governed by a stochastic differential equation. This implies that the cross-section indexing is usually endogenous to some underlying process, and is not exogenous as in the spatial-gradient approach of the model in this paper.

The paper proceeds as follows: in section 2 the social welfare problem is specified, in section 3 I solve explicitly and characterize the distributional AK version of the model, in section 4 I approximate the solution, characterize the dynamics, and perform a distributional comparative dynamics analysis for the distributional Ramsey model, and in section 5 a brief conclusion is provided.

2 The model

There is a continuum of economies indexed by $x \in \mathrm{X}$, where X is a continuum set of locations. Population is distributed along the continuum, and has a total mass normalized to one. It is postulated that set X has two features. First, it has a geometry of a ring of finite length L, as in Salop (1979).[4] Without loss of generality I assume, in most of this paper, a normalized ring with length equal to one, implying $\mathrm{X} = [-\frac{1}{2}, \frac{1}{2}]$. Second, the index space is exogenous. This assumption distinguishes our model from models in which the distribution dynamics is represented by a Fokker-Planck equation.[5]

[4] An alternative would be to choose a Riemannian manifold as in Fabbri (2016).

[5] As in mean-field control or game models.

In every location, $x \in \mathrm{X}$, an homogeneous good is produced, and can be consumed, invested locally, or shipped to other locations. Production in location x is a function of the local stock of capital, using a technology that is homogeneous across all locations. Physical capital can be reallocated. However, reallocations of capital involve a spatial friction taking the form of a diffusion-like transport adjustment cost. Savings in location x finance both local capital accumulation and net capital exports. I show this implies that, in every location x, equilibrium between demand and supply of capital is represented by a forward parabolic partial differential equation (PDE).

Savings is non-consumed income. Consumption is determined from intertemporal utility maximization by households in every location x. I assume that households are homogeneous across locations, and that there are no externalities or other market distortions. I postulate instead that there is an optimal allocation over time and across locations, of consumption and capital, that maximizes a social welfare functional, under the assumption of absence of discrimination among locations. Therefore, the social welfare functional incorporates both an exponential time discounting and a uniform spatial weighting of local utilities of consumption. The associated generalized Euler condition is a backward parabolic PDE for the marginal value of capital. This equation specifies a space-time arbitrage condition between the marginal benefit and marginal cost of capital, where the marginal benefit is equal to the local marginal productivity of capital plus an intertemporal and an inter-spatial change in the marginal value of capital. Therefore, the dual to the spatial friction in the transport of capital is a spatial wedge in the marginal value of capital inducing a spatial reallocation of savings and, therefore, of the supply of capital.

Summing up, the optimal capital allocation is governed by both pre-determined and forward-looking spatial mechanisms.[6] The drivers for the dynamics of the reallocation of capital over time are: local savings, differences in the marginal productivity of capital, the spatial friction in the transfer of capital, and the spatial wedge in the marginal value of capital.

[6] See the place of this approach in the spatial economics literature in Desmet and Rossi-Hansberg (2010).

Next I present formally the main elements of the model.

2.1 Capital accumulation constraints

2.1.1 Capital stock dynamics

The stock of capital at time $t \in \mathrm{T} = \mathbb{R}_+$ in location $x \in \mathrm{X}$ is denoted by $K(t,x)$. I assume that $K(t,\cdot) : \mathrm{X} \to \mathbb{R}_+$ is a bounded mapping, for every finite $t \in \mathrm{T}$.[7] The ensemble capital stock at time t is $\big(K(t,x)\big)_{x\in\mathrm{X}}$. For each $(t,x) \in \mathrm{T}\times\mathrm{X}$, there is an equilibrium relationship between savings, $S(t,x)$, and the sum of the instantaneous change in capital stock, $dK(t,x)/dt$, and capital depreciation. Formally, $S(t,x) = dK(t,x)/dt + \delta\, K(t,x)$, where $\delta > 0$ is the rate of capital depreciation.

While savings, $S(t,x)$, represents the supply of capital, $dK(t,x)/dt$ represents the net demand for capital in location x, at time t. In autarky, the net demand of capital is equal to the change in capital input which can be used for production, in every particular location. However, in an open economy, capital can also be reallocated to or from all particular locations within X. This means that a local imbalance between local savings and local demand of physical capital entails capital transportation across locations, such that

$$\frac{dK(t,x)}{dt} = \frac{\partial K(t,x)}{\partial t} + T(t,x), \text{ for each } (t,x) \in \mathrm{T}\times\mathrm{X}, \tag{1}$$

where $\dfrac{\partial K(t,x)}{\partial t}$ is local investment and $T(t,x)$ is the level of capital reallocation. In particular, $T(t,x)$ is the net outcome of the demand for capital located elsewhere by households located in x and of the demand for capital located in x from households located elsewhere in X. Therefore, if $T(t,x) > 0$ there will be a net capital outflow from location x, and if $T < 0$ there will be a net capital inflow to location x. I assume that the transportation of physical capital satisfies:

Assumption 1. *Consider an arbitrary region $R = [x, x+h] \subset \mathrm{X}$ with length $h > 0$. The net outflow of capital across region R is proportional to the difference between the gradients*

[7] This means that $K(t,\cdot)$ can only converge to an unbounded distribution for $t \to \infty$.

of the ensemble capital distribution evaluated at the boundaries of region R, such that

$$T(t,R) = -\tau^2 \int_{\partial R} \frac{\partial K(t,s)}{\partial x} = -\tau^2 \Big(\frac{\partial K(t,x+h)}{\partial x} - \frac{\partial K(t,x)}{\partial x} \Big) \tag{2}$$

where $\tau > 0$.

We can conceive τ^{-1} as a measure of the barriers to spatial capital flows. If τ is large (small) then the spatial adjustment cost is low (high), for any level of capital transportation. If $\tau = 0$ then we will have autarky, and a complete absence of spatial capital flows.

In the limit, the spatial capital reallocation function becomes

$$T(t,x) = \lim_{h \to 0} T(t,R) = -\tau^2 \frac{\partial^2 K(t,x)}{\partial x^2}.$$

Consequently, location x is a net capital importer (exporter) if and only if $K(t,x)$ is locally convex (concave).[8].

Therefore, the net capital flow in location x satisfies the equation[9]

$$\frac{dK(t,x)}{dt} = \frac{\partial K(t,x)}{\partial t} - \tau^2 \frac{\partial^2 K(t,x)}{\partial x^2}, \text{ for each } (t,x) \in \mathrm{T} \times \mathrm{X}. \tag{3}$$

Savings is determined by non-consumed income: $S(t,x) = Y(t,x) - C(t,x)$, where $Y(t,x)$ and $C(t,x)$ denote, respectively, output and consumption in location x at time t. The production technology is endogenous to both the capital allocation and to potential location-specific productivity heterogeneities. In order to concentrate on the endogenous generation of heterogeneity, in the rest of the paper, except in the last subsection, I assume that the technology is homogeneous across locations:

Assumption 2. *The technology of production is homogenous across locations and is represented by the production function $Y(t,x) = f\big(K(t,x)\big)$, where $f(\cdot)$ is increasing and (not necessarily strictly) concave: $f''(K) \leq 0 < f'(K)$.*

[8]For an alternative rationalization see Brito (2004) and the related literature.
[9]This is analogous to the having an analog to the Fick's law governing capital dynamics.

Summing up, the dynamics of the density $K(t,x)$ across locations and over time is governed by the quasi-linear forward parabolic partial differential equation (PDE)

$$\frac{\partial K(t,x)}{\partial t} = \tau^2 \frac{\partial^2 K(t,x)}{\partial x^2} + f\big(K(t,x)\big) - C(t,x) - \delta\, K(t,x), \text{ for each } (t,x) \in (\mathrm{T},\mathrm{X}). \quad (4)$$

Recall that spatial capital flows in this equation refer to net transportation of physical capital, not to the spatial reallocation of financial capita. We will see next that financial capital flows are driven by incentives governing the spatial allocation of saving when there are spatial differences in the marginal value of capital.

I consider the cases in which there are constant or decreasing returns to scale (CRS or DRS). There is a crucial difference between the two technologies as regards the dependence of the rate of return of capital to spatial differences in the levels of capital stock: in the CRS case the rate of return of capital is constant and space-independent, while, in the DRS case spatial differences in the stock of capital entail differences in the rate of return of capital, such that rich (poor) capital locations have a lower (higher) marginal productivity of capital. This implies that spatial reallocations of capital have different consequences in the two cases: while they leave the rate of return of capital unchanged in the CRS case, they change the rate of return of capital in the DRS case by creating an additional incentive for the spatial reallocation of financial capital.

2.1.2 Boundary conditions

The assumption that economies are located along a ring requires the introduction of the following boundary conditions

$$K\big(t, -\tfrac{L}{2}\big) = K\big(t, \tfrac{L}{2}\big), \text{ and } \frac{\partial K\big(t, -\frac{L}{2}\big)}{\partial x} = \frac{\partial K\big(t, \frac{L}{2}\big)}{\partial x}, \text{ for each } t \in \mathrm{T}, \quad (5)$$

which implies that $K(t,x)$ is a periodic function in space with period equal to L.

I also assume that the initial distribution of capital, $K(0,x)$ is fixed, and is bounded in the $L^1(\mathrm{X})$ sense. In particular, I assume that the aggregate of the ensemble capital across all locations is finite,

$$\mathsf{M}[K](0) \equiv \frac{1}{L}\int_{-L/2}^{L/2} K(0,x)\,dx < \infty. \tag{6}$$

2.2 Social welfare functional

The main issues concerning the adoption of a social welfare functional refer to the specification of individual preferences, and to the constraints that should be introduced on their aggregator such that particular social welfare criteria are satisfied.[10]

I assume that preferences are homogenous across locations, and that, in every location $x \in \mathrm{X}$, there is a representative household whose preferences are represented by a discounted intertemporal utility functional, $\mathsf{U}[C](x) = \int_t^\infty u\big(C(s,x)\big)\,e^{-\rho(t-s)}ds$, where $\rho > 0$ is the rate of time preference, and the utility function is globally increasing and concave ($u''(C) < 0 < u'(C)$ for all $C \in \mathbb{R}_+$).

We can aggregate preferences over locations by extending a Bergson-Samuelson social welfare functional to an intertemporal context.[11] A social welfare functional, consistent with several different ethical postulates, has the general form $\mathsf{W}\big[\mathsf{U}\big] = \int_X \mathsf{U}[C](x)\,n(x)\,dx$, where $n(x)$ is the utility weight ascribed by the social planner to households located in $x \in \mathrm{X}$. Several particular cases can be specified - for instance, Benthamian, Millian, von-Neumann-Morgenstern, egalitarian or Rawlsian - by choosing particular weighting functions, $n(\cdot)$, . The Benthamian social welfare functional can be represented by a location-unweighted functional $\mathsf{W}[C] = \int_{\mathrm{X}}\int_0^\infty u\big(C(t,x)\big)e^{-\rho t}\,dt\,dx$. All the other social welfare functionals involve weighting local preferences. The simpler weighting schemes involve spatial discounting or

[10] Boundedness of the social welfare functional is another matter of concern when the domain of locations, X, is unbounded.

[11] Bergson (1938) and (Samuelson, 1947, p.219-229) presented an additive social welfare functional as a sum of cardinal utility functions. Harsanyi (1955) showed that aggregate social preferences based upon ordinal utility functions, and obeying some postulates (v.g, symmetry, independence, transitivity etc), can be represented as a weighted sum of individual cardinal (or Bernoullian) utility functions. Those postulates also satisfy the two main Rawlsian criteria of impartiality and unanimity (see Mueller (2003)).

averaging. An example of spatial discounting is the von-Neumann-Morgenstern welfare functional $\mathsf{W}[C] = \int_{\mathrm{X}} \int_0^\infty u\big(C(t,x)\big)\, e^{-\rho t}\, e^{-x^2}\, dt\, dx$ is used in, e. g., Boucekkine et al. (2009). In this functional, spatial discounting penalizes locations further away from a pivotal location - in this case $x = 0$. However, this spatial discounting has two unwelcome features: it introduces a preference relation over locations, which violates Harsanyis's symmetry postulate, and tends to force rejection of a homogeneous spatial distribution as an optimal distribution in the steady state (even in the case in which the other parameters of the model are spatially homogeneous).

I assume instead an average spatial aggregation scheme, which is equivalent to assuming an uniform distribution as the spatial weighting scheme:

Assumption 3. *The social welfare functional is*

$$\mathsf{W}[C] \equiv \frac{1}{L} \int_{-L/2}^{L/2} \int_0^\infty u\big(C(t,x)\big)\, e^{-\rho t}\, dt\, dx, \tag{7}$$

where the utility function satisfies $u'(C) > 0 > u''(C)$, *and* $u'''(C) > 0$ *for any* $C \in \mathbb{R}_{++}$.

We can see this social welfare functional as an extension of the Millian criterium to an intertemporal context. It has two important properties: first, it is bounded even in the case in which the consumption distribution tends asymptotically to a location-homogeneous $L^1(\mathrm{X})$, or flat, distribution growing in time at a rate smaller than $\rho > 0$; and second, it satisfies one important welfare ethical postulate:[12]

Lemma 1. *The social welfare functional* (7) *satisfies the Pigou-Dalton transfer principle.*

This means that, in the presence of an initial heterogeneity in consumption across locations, transferring consumption from locations with higher levels of consumption to locations with lower levels of consumption is welfare enhancing.

[12] All the proofs of this paper are presented in an online Appendix.

2.3 The optimal dynamic distribution problem

The **social welfare problem** consists in finding the optimal intertemporal distributive allocations for consumption and for the capital stock, $\big(C^*(t,x), K^*(t,x)\big)_{(t,x)\in \mathrm{X}\times\mathrm{T}}$, such $C^*(t,x) > 0$ and $K^*(t,x) > 0$, for each $(t,x) \in \mathrm{T}\times\mathrm{X}$, that maximize the social welfare functional, in equation (7), subject to the capital accumulation constraint, in equation (4), and to the boundary constraints, in equation (5), given the initial distribution, in equation (6), and the terminal constraint, [13]

$$\lim_{t\to\infty} e^{-\rho t} K(t,x) \geq 0, \text{ for each } x \in \mathrm{X}. \tag{8}$$

The necessary first-order conditions are:

Proposition 1. *Assume there is an optimal allocation* $\big(C^*(t,x), K^*(t,x)\big)_{(t,x)\in \mathrm{T}\times\mathrm{X}}$*, satisfying the admissibility conditions (4), (5), (6), and (8). Then there is a positively-valued distributional co-state variable,* $Q : \mathrm{T}\times\mathrm{X} \to \mathbb{R}_{++}$*, which is a piecewise continuous function of time, such that the following conditions are satisfied:*

1. *the static optimality condition*

$$u'(C^*(t,x)) = Q(t,x), \ \textit{for each } (t,x) \in \mathrm{T}\times\mathrm{X}; \tag{9}$$

2. *the generalized Euler equation*

$$\frac{\partial Q(t,x)}{\partial t} = -\tau^2 \frac{\partial^2 Q(t,x)}{\partial x^2} + Q(t,x)\Big(\rho - r(K^*(t,x)\Big), \ \textit{for each } (t,x) \in \mathrm{T}\times\mathrm{X}; \tag{10}$$

where $r(K) = f'(K) - \delta$ *is the marginal productivity of capital net of depreciation;*

3. *the dual boundary conditions*

$$Q\big(t, -\tfrac{L}{2}\big) = Q\big(t, \tfrac{L}{2}\big), \ \textit{and} \ \frac{\partial Q\big(t,-\frac{L}{2}\big)}{\partial x} = \frac{\partial Q\big(t,\frac{L}{2}\big)}{\partial x}, \ \textit{for each } t \in \mathrm{T}; \tag{11}$$

[13] Mathematically, this is an infinite horizon discounted optimal average control problem for a parabolic PDE defined over a ring.

4. and the transversality condition

$$\lim_{t\to\infty} Q(t,x)\,K^*(t,x)\,e^{-\rho t} = 0,\ \textit{for each}\ x \in \mathrm{X}. \tag{12}$$

The static optimality condition (9), is a standard local static instantaneous arbitrage condition between the marginal benefit and marginal cost of consumption, where the first is measured by local marginal utility of consumption and the second is measured by the marginal value of capital. The generalized Euler equation (10) introduces an additional local intertemporal and inter-spatial arbitrage condition between the marginal benefit and the marginal cost of capital. In this case, the marginal cost of capital is measured by the rate of time preference, ρ, which is time- and location-invariant. The marginal cost of capital is measured by the sum of three time- and location-dependent components: the net rate of return of capital, $r(t,x)$, plus a time wedge and a spatial wedge in the value of capital.

As it is a non-standard Keynes-Ramsey rule, we can write equation (10) as

$$\Big(\frac{\partial Q(t,x)}{\partial t} + V(t,x)\Big)\,\frac{1}{Q(t,x)} + r(t,x) = \rho,\ \text{for each}\ (x,t) \in \mathrm{X} \times \mathrm{T}, \tag{13}$$

where the time wedge is represented by the rate of change in the value of capital stock in location x, $\frac{\partial Q(t,x)/\partial t}{Q(t,x)}$, and the spatial wedge is represented by the relative value of relocating savings to other locations, $V(t,x)/Q(t,x)$, a kind of earnings-price ratio. To interpret the last component, we can think of function $V(t,x)$ as a dual to function $T(t,x)$. Consider again a region $R = [x, x+h]$, and assume that the earnings from relocating financial capital are proportional to the difference in the gradient of the ensemble capital value distribution at the two boundaries of region R,

$$V(t,R) = \tau^2 \int_x^{x+h} \frac{\partial Q(t,x)}{\partial x}.$$

Taking the limit, we find the earnings for a particular location $x \in \mathrm{X}$,

$$V(t,x) = \lim_{h \to 0} V(t,R) = \tau^2 \frac{\partial^2 Q(t,x)}{\partial x^2}.$$

Therefore, if the spatial wedge is proportional to the local curvature of the ensemble distribution of the marginal value of capital, $\big(Q(t,x)\big)_{x \in \mathrm{X}}$, then earnings from relocating capital, V, are negative (positive), and moving savings represents a cost (benefit), if $Q(t,x)$ is locally concave (convex).

The Keynes-Ramsey rule is the driver for the forward looking behavior of consumption, and therefore, of saving. In the a-spatial AK or Ramsey models this rule only considers the time wedge. In our case, if, in a particular location, the rate of time preference is greater (smaller) than rate of return of capital there will only exist an increase (decrease) in the marginal value of capital, and therefore in the utility of consumption, if the spatial wedge does not dominate the time wedge. It is possible that a strong spatial wedge may generate an incentive for a type of behavior which cannot occur in the standard a-spatial models. For instance, it is possible that, if the rate of return of capital is much larger than the rate of time preference, the increase in savings, generated by that imbalance, will not be invested in the local financial assets, and therefore in local capital formation, but instead in financial assets that finance capital located elsewhere.

The optimal rule for spatial reallocation of consumption, associated to the maximization of social welfare, can be more clearly seen when we consider both equations (9) and (10) to write the Euler equation as

$$\frac{\partial C(t,x)}{\partial t} + \tau^2 \left(\frac{\partial^2 C(t,x)}{\partial x^2} - \frac{\pi(t,x)}{C(t,x)} \left(\frac{\partial C(t,x)}{\partial x} \right)^2 \right) = \frac{C(t,x)}{\theta(t,x)} \big(r(t,x) - \rho\big),$$

where $\theta(t,x) \equiv -\frac{u''\left(C(t,x)\right) C(t,x)}{u'\left(C(t,x)\right)} > 0$ and $\pi(t,x) \equiv -\frac{u'''\left(C(t,x)\right) C(t,x)}{u''\left(C(t,x)\right)} > 0$.

The total change in consumption is the sum of the change in autarky plus the spatial reallocation of consumption that is optimal from a social welfare perspective. Now, we see that the optimal spatial reallocation of consumption involves both a backward diffusion mechanism, similarly to the spatial wedge for Q, and a non-linear transportation term. The first generates a reallocation of consumption towards location x if the ensemble distribution $\big(C(t,x)\big)_{x\in \mathrm{X}}$ is locally concave, and an ambiguously signed net reallocation if it is locally convex. The last term shows there is mass reallocation of consumption across locations seeking to homogenize its distribution.

Summing up, an optimal allocation $\big(C^*(t,x), K^*(t,x)\big)_{(t,x)\in \mathrm{T}\times\mathrm{X}}$ can be obtained, or at least characterized, from the solutions of the distributed maximized Hamiltonian dynamic system (DMHDS),

$$
\begin{aligned}
\frac{\partial K(t,x)}{\partial t} &= \tau^2 \frac{\partial^2 K(t,x)}{\partial x^2} + f(K(t,x)) - C(t,x) - \delta K(t,x), \text{ for } (t,x) \in \mathrm{T}\times\mathrm{X}, \quad (14)\\
\frac{\partial C(t,x)}{\partial t} &= -\tau^2 \left(\frac{\partial^2 C(t,x)}{\partial x^2} - \frac{\pi(t,x)}{C(t,x)} \Big(\frac{\partial C(t,x)}{\partial x}\Big)^2 \right) + \\
&\quad + \frac{C(x,t)}{\theta(t,x)} \Big(f'(K(t,x)) - \rho - \delta \Big), \text{ for } (t,x) \in \mathrm{T}\times\mathrm{X}, \quad (15)
\end{aligned}
$$

together with the boundary conditions for capital (5) and for consumption,

$$
C\Big(t,-\tfrac{L}{2}\Big) = C\Big(t,\tfrac{L}{2}\Big) \; \frac{\partial C\big(t,-\frac{L}{2}\big)}{\partial x} = \frac{\partial C\big(t,\frac{L}{2}\big)}{\partial x}, \text{ for each } t \in \mathrm{T}, \quad (16)
$$

the transversality condition,

$$
\lim_{t\to\infty} u'(C(t,x))\, K(t,x)\, e^{-\rho t} = 0, \text{ for each } x \in \mathrm{X}, \quad (17)
$$

and the initial condition (6). The problem is well posed if we can prove the existence of two mappings $K : \mathrm{T}\times\mathrm{X} \to \mathbb{R}_{++}$ and $C : \mathrm{T}\times\mathrm{X} \to \mathbb{R}_{++}$ that satisfy equations (14)-(17).

3 A distributional AK model

In this section I assume a spatially-homogeneous constant returns to scale technology, and therefore, a linear production function $Y(t,x) = A\,K(t,x)$.[14] This assumption implies that differences in the marginal productivity of capital cannot generate incentives for savings reallocations across locations.

Standard a-spatial endogenous growth theory suggests that a generalized or distributional balanced growth path (BGP) can exist. However, several specific questions, related to the distributional feature of our model, arise: is unbounded growth admissible, in the sense of satisfying the generalized transversality condition ? Would we have transitional dynamics converging to the BGP, or is the solution coincident with a BGP, as in the benchmark a-spatial model ? Is the distributional BGP spatially homogeneous or not ? We prove next that, under weak conditions, the answers to those three questions are affirmative.

Assuming an isoelastic utility function $u(C) = \big(C^{1-\theta}-1\big)/\big(1-\theta\big)$ for $\theta \geq 1$, we obtain the following specification for the DMHDS, in equations (14) and (15), for each $(t,x) \in \mathrm{T} \times \mathrm{X}$,

$$\frac{\partial K(t,x)}{\partial t} = \tau^2 \frac{\partial^2 K(t,x)}{\partial x^2} + r\,K(t,x) - C(t,x), \tag{18}$$

$$\frac{\partial C(t,x)}{\partial t} = -\tau^2 \left[\frac{\partial^2 C(t,x)}{\partial x^2} - \frac{1+\theta}{C(t,x)}\left(\frac{\partial C(t,x)}{\partial x}\right)^2\right] + \gamma\,C(t,x), \tag{19}$$

where r is the net total factor productivity and γ is the long run growth rate,

$$r \equiv A - \delta,\ \gamma \equiv \frac{r-\rho}{\theta}. \tag{20}$$

Observe that these two expressions are formally identical to the analogues in the standard

[14] As in Frankel (1962). The benchmark a-spatial AK model is presented in Rebelo (1991). Several versions of the spatial AK model have been presented in Brito (2004), Boucekkine et al. (2009), Boucekkine et al. (2013), Fabbri (2016) and Boucekkine et al. (2019b). In this section I solve the model using classic Fourier transform methods.

a-spatial AK model, The ttransversality condition (17) becomes

$$\lim_{t\to\infty} e^{-\rho t}\, K(t,x)\, C(t,x)^{-\theta} = 0, \text{ for each } x \in \mathrm{X}. \tag{21}$$

In the next section 3.1 I explicitly solve this problem and characterize its the solution, when the spatial domain is a ring of finite length $L = 1$. In section 3.2 I show that those results can be extended to an unbounded domain, if the initial distribution is bounded in $L^1(\mathbb{R})$.

3.1 Bounded domain

In this section I assume that the location domain is a ring with a finite normalized length $L = 1$, implying $\mathrm{X} = [-\frac{1}{2}, \frac{1}{2}]$, and that the initial distribution of the capital stock over the ring, $k_0(x)$, is a periodic function in the $L^1(\mathrm{X})$ space, satisfying

$$\begin{cases} k_0\big(\text{-}\frac{1}{2}\big) = k_0\big(\frac{1}{2}\big), \ \dfrac{\partial k_0\big(\text{-}\frac{1}{2}\big)}{\partial x} = k_0 \dfrac{\partial k_0\big(\frac{1}{2}\big)}{\partial x}, \\ \mathsf{M}[K](0) = \int_{\text{-}1/2}^{1/2} k_0(x)\, dx < \infty. \end{cases} \tag{22}$$

We obtain an explicit solution for the optimal allocation problem:

Proposition 2. *Assume that $\theta \geq 1$ and $r > \gamma$ and let the initial distribution $k_0(x)$ satisfy conditions (22). Then the coupled system (18)-(19), satisfying boundary conditions (5) and (16), for $L = 1$, and the transversality condition (21), has the closed form solution*

$$K(t,x) = \int_{\text{-}1/2}^{1/2} k_0(y)\, g_k(t, x-y)\, dy, \ \textit{for each } (t,x) \in \mathrm{T} \times \mathrm{X}, \tag{23}$$

$$C(t,x) = \int_{\text{-}1/2}^{1/2} k_0(y)\, g_c(t, x-y)\, dy, \ \textit{for each } (t,x) \in \mathrm{T} \times \mathrm{X}, \tag{24}$$

with kernels

$$g_k(t,x) = \sum_{\omega\in\mathbb{Z}} e^{\lambda_\omega t + i\,2\pi\omega x}$$
$$g_c(t,x) = \sum_{\omega\in\mathbb{Z}} \psi_\omega\, e^{\lambda_\omega t + i\,2\pi\omega x}$$

where $\psi_\omega \equiv r - \gamma + (\theta - 1)\,\xi_\omega$, *and* $\mathbb{Z} = \{0, \pm1, \pm2, \dots\}$ *is the set of integers and*

$$\lambda_\omega \equiv \gamma - \theta\,\xi_\omega,\ \textit{where}\ \xi_\omega \equiv \big(2\pi\tau\omega\big)^2\ \textit{for}\ \omega \in \mathbb{Z}. \tag{25}$$

From the solution for the capital stock, in equation (23), we can obtain the solution for output, as $Y(t,x) = A\,K(t,x)$, and the solution for savings is $S(t,x) = Y(t,x) - C(t,x)$, where consumption is in equation (24).

Analysing equations (23) and (24) we readily observe that the time evolution depends on the kernels $g_k(t,x)$ and $g_c(t,x)$. These kernels are weighed sums of the spectral growth factors $\{e^{\lambda_\omega t}\}_{\omega\in\mathbb{Z}}$, where the rate of growth for any frequency $\omega \in \mathbb{Z}$, λ_ω, is defined in (25).

Function λ_ω is the difference between a positive, frequency-independent, constant γ, and the product of the inverse of the elasticity of intertemporal substitution, θ, with the wave number (or Fourier mode) $\xi_\omega = \big(2\pi\tau\omega\big)^2$. The wave number ξ_ω expresses the rate of propagation of shocks along a particular frequency $\omega \in \mathbb{Z}$. It increases with the inverse of the costs of transportation, τ, and is an increasing and convex mapping of the frequency space into the set of non-negative real numbers, $\xi : \mathbb{Z} \to \mathbb{R}_+$. It has a minimum, $\xi_0 = 0$, for $\omega = 0$, for any value of τ, or for $\tau = 0$, for any value of ω. Therefore, the maximum λ_ω is equal to γ, and is attained for $\tau = 0$ or for $\omega = 0$, and $\tau > 0$. Those two cases correspond to two different environments: the first is related to autarky, in which there is no spatial reallocation of capital, and the second corresponds to an open economy, in which capital can be reallocated. Therefore, the dynamics of capital reallocation is determined by the spectra of frequencies.

Figure 1 depicts λ_ω as a function of frequencies ω, for an open economy environment. As we can see, it is a concave function with a maximum $\lambda_0 = \gamma > 0$, for frequency $\omega = 0$, and has $\lim_{|\omega|\to\infty} \lambda_\omega = -\infty$. We will see that λ_0 is the asymptotic growth rate of the ensemble distribution of capital and consumption. As it is positive we have spectral instability, meaning that the whole distribution will become unbounded asymptotically over time. This is a necessary condtion for the existence of long-run growth, or a balanced growth path (BGP) with positive growth rates. We also observe that increases in γ will move the whole spectrum $\{\lambda_\omega\}_{\omega\in\mathbb{Z}}$ up, leading to increases in the long-run growth rate.

In Figure 1 we also see that λ_ω is positive for a finite subset of frequencies ω close to zero and its is negative for frequencies higher that a small absolute value for ω.[15] While the first set of frequencies generates spectral instability, in the sense that $e^{\lambda_w t}$ will be increasing in time, the second set of frequencies generates spectral stability, in the sense that $e^{\lambda_w t}$ tend asymptotically to zero. The convergence to zero at a higher rate implies that the spatial difference in the distribution of capital (and consumption) diminishes at a faster rate. In other words, this entails a faster spatial diffusion of capital by reducing more vigorously the spatial concentration of capital and consumption. As can be seen Figure 1, higher values for τ and for θ will generate a flatter distribution along the generalized transition path. While the first effect is related to the reduction in the spatial frictions on the spatial reallocation of capital, the second is associated to the workings of the Pigou-Dalton effect, that is to a higher social preference for a smoother distribution of consumption across locations. However, differently from the spatial barrier's parameter, τ, decreases in the elasticity of intertemporal substitution, $1/\theta$, combines the two types of effects: it reduces the asymptotic rate of growth γ, by permanently reducing aggregate savings, and increases the diffusive feature of the spatial propagation mechanism.[16]

`Figure 1 around here`

Another conclusion can be drawn from the time behavior of the two kernels: if the initial distribution $k_0(x)$ is not flat, then the adjustment to the BGP is not instantaneous, as in the

[15] Defining $|\omega_0| = \frac{1}{2\pi\tau}\sqrt{\frac{\gamma}{\theta}}$, it easy to show that eigenfunctions are positive for $\{\omega \in \mathbb{Z} : |\omega| < |\omega_0|\}$, and are negative for $\{\omega \in \mathbb{Z} : |\omega| > |\omega_0|\}$.

[16] Therefore a smaller $|\omega_0|$ entails a stronger diffusive process.

a-spatial AK model. However, I can provide some general characterization of the dynamics and distributional features of the optimal allocation of capital.

First, a conservation law is satisfied: given an initial aggregate for the ensemble capital stock, $\mathsf{M}[K](0)$, the aggregate for the ensemble capital stock for any posterior point in time is equal to the initial aggregate times a growth factor, with the rate of growth equal to $\lambda_0 = \gamma$,[17]

$$\mathrm{M}[K](t) = \int_{-1/2}^{1/2} K(t,x)\,dx = e^{\gamma t}\,\mathrm{M}[K](0), \text{ for each } t \in \mathrm{T}. \tag{26}$$

This means that the solution follows an ensemble balanced growth path (BGP), or the solution is a BGP in the $L^1(\mathrm{X})$ sense. This property can occur both in an open economy or in autarky, because the BGP growth rate in the a-spatial AK model is equal to γ.

Second, the capital stock in an open economy satisfies, $K(t,x) \neq k_0(x)\,e^{\gamma t}$, almost everywhere in X, which is different from the case of autarky, where $K(t,x) = k_0(x)\,e^{\gamma t}$, for all $x \in \mathrm{X}$. Therefore, the capital stock distribution does not follow a BGP because there is spatial redistribution of capital among locations, which does not occur in autark. Third, the spatial distribution of capital stock converges asymptotically towards a common, spatial-homogeneous, BGP with positive growth rate $\gamma > 0$[18]

$$\lim_{t\to\infty} K(t,x) = \mathsf{M}[K](0)\,e^{\gamma t}. \tag{27}$$

We can show that consumption and savings converge to spatially-homogeneous distributions of consumption and savings, as well,

$$\lim_{t\to\infty} C(t,x) = (r-\gamma)\,\mathsf{M}[K](0)\,e^{\gamma t}, \text{ and } \lim_{t\to\infty} S(t,x) = \gamma\,\mathsf{M}[K](0)\,e^{\gamma t}.$$

From the previous properties, we can conclude that in an open environment (where $\tau > 0$)

[17] See Corollary 1 in the Appendix for the proof.

[18] See Corollary 2 in the Appendix for the proof.

we have, asymptotically,

$$\lim_{t\to\infty} \frac{K(t,x)e^{-\gamma t}}{k_0(x)} \gtreqless 1, \text{ if } k_0(x) \lesseqgtr \mathsf{M}[K](0), \text{ for } x \in \mathrm{X}. \tag{28}$$

There is thus a spatially-induced transition dynamics towards a common BGP, featuring a redistribution of capital from initially capital rich locations towards initially capital poor locations. This process is achieved not only by the transportation of physical capital but also relocation of savings, which results from social welfare enhancing redistribution of consumption across space. Convergence to a spatially-homogeneous distribution is possible because initially capital-poor locations have, at some point in time, a higher level of savings, generated locally or by financial capital transfers from elsewhere, and long-run growth is only possible because detrended long run savings, $\lim_{t\to\infty} e^{-\gamma t} S(t,x)$, is positive for every location.

`Figure 2 around here`

To illustrate the dynamics in our model, I consider an initial Laplace distribution (see Benhabib and Hager (2021))

$$k_0(x) = \beta \frac{e^{-2\beta\,|x|}}{1-e^{-\beta}}, \text{ for } \beta > 1, \tag{29}$$

that is normalized to satisfy $\mathrm{M}[K(0)] = 1$. This distribution features an initially capital-rich central location, around $x = 0$, and an initially capital-poor peripheral location, around $|x| = \frac{1}{2}$. Figure 2 illustrates the distributional dynamics, with and without trends, for capital, in Subfigures 2a. for consumption, in Subfigures 2b, and for savings, in Subfigures 2c.

At time $t = 0$, an initial heterogeneous capital distribution implies an initial heterogeneous income distribution, which is greater for the center than for the periphery. As the ensemble capital distribution is convex, there is a net capital outflow from the periphery to the center (see our previous discussion concerning equation (3)). In addition, there is an incentive for a relative high consumption in the center than in the periphery, because the former location anticipates a relatively higher level of income over time (see Subfigure 2b). Both effects imply that local savings in the center will be smaller than in the periphery (see Subfigure 2c). This causes a lower level of capital accumulation in the center location.

Over time, this process is rapidly reversed. There is, at the center, a smaller increase in consumption, an increase in savings, a transformation from capital importer to capital exporter, and a permanent investment in local capital from domestic savings over time. At the periphery savings declines and there are increases capital formation mostly coming from central locations. Looking at the detrended distribution of capital we see that the initial convexity turns into ensemble concavity fading away over time, meaning that capital starts agglomerating towards the centre but quickly changes direction by progressively diffusing away from the center, until it becomes asymptotically evenly distributed in the long-run. This redistribution of capital, which I proved in (28), can be clearly illustrated in Figure 3.

The left diagrams in Subfigures 2a and 2b illustrate our previous generic conclusions that there is a convergence of all variables to a flat BGP growing at a positive growth rate γ. This is possible because, as shown in Subfigure 2c, savings is positive across all locations in the long-run.

`Figure 3 around here`

3.2 Unbounded domain

The above results apply to the case in which $L = \infty$ and $X = \mathbb{R}$, if the initial distribution satisfies

$$\begin{cases} \lim_{x\to-\infty} k_0(x) = \lim_{x\to\infty} k_0(x), \text{ and } \lim_{x\to-\infty} \frac{\partial k_0(x)}{\partial x} = \lim_{x\to\infty} \frac{\partial k_0(x)}{\partial x} \\ \mathsf{M}[K](0) = \int_{-\infty}^{\infty} k_0(x)\, dx < \infty. \end{cases} \tag{30}$$

This allows us to obtain a more specific (formal) closed form solution involving Gaussian kernels:[19]

Proposition 3. *Assume an extension to the previous social welfare problem where* $L \to \infty$ *with the initial distribution* $k_0(x)$ *satisfying conditions* (15). *Then the problem has a (formal) closed form solution that takes the form of an exponential trend multiplied by a detrended time varying Gaussian distribution*

$$K(t,x) = e^{\gamma t}\, k(t,x), \text{ and } C(t,x) = e^{\gamma t}\, c(t,x), \text{ for each } (t,x) \in \mathrm{T} \times \mathrm{X}$$

where the detrended distribution functions for capital and consumption are

$$k(t,x) = \int_{-\infty}^{\infty} k_0(\xi)\, \frac{e^{-\frac{(x-\xi)^2}{2\,\sigma(t)^2}}}{\sqrt{2\pi}\,\sigma(t)}\, d\xi, \text{ for } (t,x) \in \mathrm{T} \times \mathrm{X}, \tag{31}$$

$$c(t,x) = \int_{-\infty}^{\infty} k_0(\xi)\, \psi(t, x-\xi)\, \frac{e^{-\frac{(x-\xi)^2}{2\,\sigma(t)^2}}}{\sqrt{2\pi}\,\sigma(t)}\, d\xi, \text{ for } (t,x) \in \mathrm{T} \times \mathrm{X} \tag{32}$$

where the time-varying standard deviation is $\sigma(t) \equiv \tau\sqrt{2\,\theta\,t} \geq 0$, *and*

$$\psi(t,x) \equiv r - \gamma + (\theta - 1)\,\tau^2 \left(\frac{\sigma(t)^2 - x^2}{\sigma(t)^4} \right). \tag{33}$$

We say this is a formal solution because, although it satisfies all first order necessary conditions, we still have to verify that $\mathbb{R}_+$ is the range of function $c(t,x)$, in equation (32). The assumption that k_0 is a mapping $\mathrm{X} \to \mathbb{R}_{++}$ is a necessary but not a sufficient condition for that property. Equation (18) shows us that the magnitude of the elasticity parameter θ is at the origin of the problem: for a logarithmic utility function, $\theta = 1$, the range of $c(\cdot, x)$ is $\mathbb{R}_{++}$, but if $\theta > 1$ this may not be the case. However, the fast decay to zero of the Gaussian exponential will force this second term to be very small, and to decrease over time to zero.

[19] This is well known from Brito (2011), and Boucekkine et al. (2013). Our approach can be justified by the recent developments concerning the characterization of the solution of the heat equation as in, for instance, Vázquez (2017).

Both distributions (31) and (32) provide a direct formal confirmation for our previously stablished bounded-space dynamics, that the distribution dynamics has a diffusion-like characteristic, and that the speed of convergence to a flat distribution is higher for greater values of τ and θ. In this case this is directly translated in the time evolution of the standard deviation of the Gaussian exponential, $\sigma(t)$: it increases over time, it is unbounded asymptotically, $\lim_{t\to\infty} \sigma(t) = \infty$, and, its rate of variation in time increases both with τ and θ.

In particular, we can show that the detrended distribution, for this unbounded domain case, have the same asymptotic properties as for the bounded case: the detrended capital stock distribution, for any $x \in (-\infty, \infty)$ tends, asymptotically to the aggregate initial capital

$$\lim_{t\to\infty} k(t,x) = \lim_{t\to\infty} \int_{-\infty}^{\infty} k_0(y)\, g_k^d(t, x-y)\, dx = \mathsf{M}[K](0)$$

and the detrended consumption distribution, for any $x \in (-\infty, \infty)$ tends, asymptotically to

$$\lim_{t\to\infty} c(t,x) = \lim_{t\to\infty} \int_{-\infty}^{\infty} k_0(y)\, g_c^d(t, x-y)\, dx = (r-\gamma)\, \mathsf{M}[K](0).$$

3.3 Growth facts

Therefore, the main growth facts which we can extract from the distributional AK-model are: (1) there is a BGP growing at rate γ, which is the same as in a related a-spatial AK-model; (2) however, differently from the a-spatial AK-model, that BGP is only reached asymptotically and it this the same for all locations, inependently from the initial level of the local capital stock; and (3) the transitional dynamics is not only a consequence of the existence of transport frictions, in the spatial reallocation of capital, but it is also a consequence of the existence of a spatial wedge in the marginal value of capital inducing a reallocation of consumption and savings. Therefore, in our distributional-AK model there is a distributionally-induced transition dynamics, in the sense that the relatively slow elimination of spatial heterogeneity is the source of transitional dynamics in the convergence of the distributions of capital and consumption towards a homogeneous BGP.

4 A distributional Ramsey model

In this section I present a distributional Ramsey model in which the technology displays decreasing returns to scale (DRS), not constant returns to scale (CRS) as in the distributional AK model.[20] This assumption implies that two locations having different capital intensities will have different marginal returns to capital, and this difference creates a new endogenous driver for the spatial reallocation of capital. In particular, locations with lower (higher) capital intensities, as they also have higher (lower) marginal productivities of capital, are endowed with a positive (negative) incentive for attracting financial capital, in addition to the other time and spatial wedges already presented in the distributional AK model.

The DMHDS I derived in Proposition 1, compactly written in the space (K, Q),[21] is

$$\frac{\partial K(t,x)}{\partial t} = \tau^2 \frac{\partial^2 K(t,x)}{\partial x^2} + f\left(K(t,x)\right) - (u')^{-1}\big(Q(t,x)\big) - \delta\, K(t,x), \text{ for } (t,x) \in \mathrm{T} \times \mathrm{X} \quad (34)$$

$$\frac{\partial Q(t,x)}{\partial t} = -\tau^2 \frac{\partial^2 Q(t,x)}{\partial x^2} + Q(t,x)\left(\rho + \delta - f'\left(K(t,x)\right)\right), \text{ for } (t,x) \in \mathrm{T} \times \mathrm{X} \quad (35)$$

where the production function is non linear and satisfying satisfies $f''(K) < 0 < f'(K)$. The rate of return for capital is $r(K) = f'(K)$. In this section I assume throughout that the domain of locations is again a ring of finite size $L = 1$. Therefore $\mathrm{X} = [-\frac{1}{2}, \frac{1}{2}]$.

Differently from the CRS case, this quasi-linear parabolic planar PDE does not have a closed form solution. However, as I assume that utility and the production functions are cross-sectionally homogeneous Inada functions, and the location domain is bounded, we can study the qualitative dynamics by performing a linear approximation in the neighborhood

[20]The original seminal contribution is Ramsey (1928). A standard version has been stablished in Cass (1965) and Koopmans (1965). In Brito (2004) we presented a preliminary version of this spatial, or distributional, Ramsey model.

[21]We use this equivalent representation, instead of (K, C), for analytical convenience.

of a flat steady state.[22]

In the next subsection 4.1 I show that spatial-heterogeneity introduces a new type of distributional dynamics, which is not present both in the a-spatial Ramsey model and in the distributional AK model, which can be analysed if we distinguish stability in a distribution sense and stability in the aggregate (L^1) sense.

In subsection 4.2 I develop a comparative dynamics analysis for a spatially-heterogeneous non-anticipated and permanent change in productivity, starting from a flat steady state. I show that capital and consumption converge to a heterogeneous ergodic long run distributions in which there are permanent local imbalances in the current accounts.

4.1 Linearized system dynamics

If the production and the utility functions have the Inada property, then there is a unique non-singular steady state $(K(x), Q(x)) = (K^*, Q^*)$, for every $x \in \mathrm{X}$, where $K^* = (f')^{-1}(\rho+\delta)$ and $Q^* = u'\big(f(K^*) - \delta\, K^*\big)$.

If we introduce a small variation, denoted by $\big(\tilde{K}(t,x), \tilde{Q}(t,x)\big)_{(t,x)\in \mathrm{T}\times\mathrm{X}}$, in the neighborhood of a spatially-homogeneous stationary, or flat, distribution, then the perturbed level and marginal value of the capital stock are, respectively, $K(t,x) = K^* + \tilde{K}(t,x)$ and $Q(t,x) = Q^* + \tilde{Q}(t,x)$. Linearizing the system (34)-(35) in the neighborhood of the homogeneous distribution, yields a variational linear forward-backward PDE in $(\tilde{K}, \tilde{Q})$

$$\frac{\partial \tilde{K}(t,x)}{\partial t} = \tau^2\, \frac{\partial^2 \tilde{K}(t,x)}{\partial x^2} + \rho\, \tilde{K}(t,x) - \big(u''(Q^*)\big)^{-1}\, \tilde{Q}(t,x), \text{ for } (t,x) \in \mathrm{T}\times\mathrm{X}, \quad (36)$$

$$\frac{\partial \tilde{Q}(t,x)}{\partial t} = -\tau^2 \frac{\partial^2 \tilde{Q}(t,x)}{\partial x^2} - Q^*\, f''(K^*)\, \tilde{K}(t,x), \text{ for } (t,x) \in \mathrm{T}\times\mathrm{X}, \quad (37)$$

[22]This approach cannot be used if the set X is unbounded as was rightly pointed by a referee. We use a heuristic geometric approach as in Henry (1981), and (Kuehn, 2019, ch. 8). This approach is commonly used, for example, in the pattern formation literature - see van Saarloos (2003) and Cross and Greenside (2009).

whose particular solutions satisfy several constraints: they are periodic,

$$\begin{aligned}
&\tilde{K}\big(t,\tfrac{1}{2}\big)=\tilde{K}\big(t,-\tfrac{1}{2}\big), \text{ and } \frac{\partial \tilde{K}\big(t,\frac{1}{2}\big)}{\partial x}=\frac{\partial \tilde{K}\big(t,-\frac{1}{2}\big)}{\partial x}, \text{ for each } t\in \mathrm{T}\\
&\tilde{Q}\big(t,\tfrac{1}{2}\big)=\tilde{Q}\big(t,-\tfrac{1}{2}\big), \text{ and } \frac{\partial \tilde{Q}\big(t,\frac{1}{2}\big)}{\partial x}=\frac{\partial \tilde{Q}\big(t,-\frac{1}{2}\big)}{\partial x}, \text{ for each } t\in \mathrm{T};
\end{aligned}\tag{38}$$

the initial variation, $\tilde{K}(0,x)=k_0(x)-K^*$, is an even function $\tilde{K}(0,x)=\tilde{K}(0,-x)$ for $|x|\in[0,\frac{1}{2}]$ such that

$$M[\tilde{K}](0)=\int_{-1/2}^{1/2}\tilde{K}(0,x)\,dx<\infty,\tag{39}$$

and the transversality condition hold, s

$$\lim_{t\to\infty} e^{-\rho t}\left(Q^*\tilde{K}(t,x)+K^*\tilde{Q}(t,x)\right)=0, \text{for each } x\in \mathrm{X}.\tag{40}$$

In the rest of this section several results depend on the following non-negative constant

$$\mu=\mu\big(K^*,Q^*\big)\equiv\frac{f''(K^*)}{u''\big(C(Q^*)\big)}\,u'\big(C(Q^*)\big)\geq 0,\tag{41}$$

that can be understood as the marginal utility of consumption weighted by the relative concavities of technology and preferences.

Proposition 4. *The particular solution of the of PDE system, (36)-(37), satisfying conditions (38), (39), and (40) is*

$$\tilde{K}(t,x) = \int_{-1/2}^{1/2}\tilde{K}(0,y)\,g_k\big(t,x-y\big)\,dy,\ \textit{for } (t,x)\in\mathrm{T}\times\mathrm{X},\tag{42}$$

$$\tilde{Q}(t,x) = \int_{-1/2}^{1/2}\tilde{K}(0,y)\,g_q\big(t,x-y\big)\,dy,\ \textit{for } (t,x)\in\mathrm{T}\times\mathrm{X},\tag{43}$$

with kernels

$$g_k(t,x) = \sum_{\omega\in\mathbb{Z}} e^{\lambda_\omega^- t + i2\pi\omega x}, \tag{44}$$

$$g_q(t,x) = u''(Q^*) \sum_{\omega\in\mathbb{Z}} \left(\lambda_\omega^+ - \xi_\omega\right) e^{\lambda_\omega^- t + i2\pi\omega x}, \tag{45}$$

where ξ_ω is defined in equation (25), and

$$\lambda_\omega^- = \frac{\rho}{2} - \sqrt{\Delta_\omega} \equiv \frac{\rho}{2} - \left(\left(\frac{\rho}{2} - \xi_\omega\right)^2 + \mu\right)^{\frac{1}{2}},\ \omega \in \mathbb{Z}, \tag{46}$$

$$\lambda_\omega^+ = \frac{\rho}{2} + \sqrt{\Delta_\omega} \equiv \frac{\rho}{2} + \left(\left(\frac{\rho}{2} - \xi_\omega\right)^2 + \mu\right)^{\frac{1}{2}},\ \omega \in \mathbb{Z}, \tag{47}$$

where $\lambda_\omega^- + \lambda_\omega^+ = \rho > 0$ and $\lambda_\omega^+ > \xi_\omega$ for any $\omega \in \mathbb{Z}$.

The solutions for consumption, income, and savings can be obtained from $\tilde{C}(t,x) = \tilde{Q}(t,x)/u''(C^*)$, $\tilde{Y}(t,x) = (\rho+\delta)\tilde{K}(t,x)$, and $\tilde{S}(t,x) = \tilde{Y}(t,x) - \tilde{C}(t,x)$, respectively.

4.1.1 Qualitative distributional dynamics

Equations (42)-(43) show us that, analogously to the distributional AK model, the dynamics of the solutions is determined by the set of growth factors $\left\{e^{\lambda_\omega^- t}\right\}_{\omega\in\mathbb{Z}}$, which depend on the spectra of the "stable" eigenfunction $\{\lambda_\omega^-\}_{\omega\in\mathbb{Z}}$.

The "stable" eigenfunction, in equation (46), has the following properties:

1. it is negative for frequency $\omega = 0$,

$$\lambda_0^- = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}\right)^2 + \mu} < 0$$

2. for intermediate values of frequencies, two cases are possible: (a) if $\frac{\rho}{2} < \sqrt{\mu}$ then $\lambda_\omega^- < 0$ for all frequencies $\omega \in \mathbb{Z}$; or (b) if $\frac{\rho}{2} \geq \sqrt{\mu}$ then the sign of λ_ω^- is ambiguous;

3. it is negative for very large frequencies, in absolute value,

$$\lim_{\omega \to \mp\infty} \lambda_\omega^- = -\infty, \text{ because } \lim_{\omega \to \mp\infty} \xi_\omega = +\infty.$$

Comparing the eigenfunction (46) with the analog for the distributional AK model, in equation (25), we see that the difference in the spectral distribution is related to frequencies closer to $\omega = 0$.

First, the fact that now λ_0^- is negative implies that there is convergence in time to some type of stationary long-run distribution, instead of permanent long-run growth as in the AK model. This dynamic property is a consequence of the existence of decreasing returns to scale. Furthermore, this result, that also holds for $\tau = 0$, is also consistent with the saddle-point stability of the benchmark a-spatial Ramsey model, and with the feature, already discussed for the a-spatial AK model, that an initial heterogeneity in the distribution of capital will not be immediately dissipated.

Second, when we have an open economy environment, i.e, when $\tau > 0$, the potential existence of spectral instability for small non-zero frequencies is a new feature of the present model. There are two possibilities:

1. if λ_ω^- is negative for all frequencies $\omega \in \mathbb{Z}$ then $\lim_{t\to\infty} e^{\lambda_\omega^- t} = 0$ for all frequencies and we say we have spectral stability, in the sense that distributions $\tilde{K}(t,x)$ and $\tilde{Q}(t,x)$ will decay to zero asymptotically for all locations $x \in \mathrm{X}$. In this case, the initial steady state will be reached back asymptotically, for any initial perturbation $k_0(x)$;

2. if λ_ω^- is positive for a subset of frequencies $\mathbb{Z}^u \subset \mathbb{Z}$ then there will be a set of growth factors converging to zero, $\lim_{t\to\infty} e^{\lambda_\omega^- t}\Big|_{w\in\mathbb{Z}^s} = 0$, and another set of growth factors that are unbounded in time, $\lim_{t\to\infty} e^{\lambda_\omega^- t}\Big|_{w\in\mathbb{Z}^u} = \infty$. In this case, as the eigenfunction for $\omega = 0$, λ_0^-, is negative some type of permanent (but ergodic) spatial pattern emerges.

A sufficient condition for spectral stability is $\frac{\rho}{2} < \sqrt{\mu}$, and a necessary condition for spectral instability is $\frac{\rho}{2} \geq \sqrt{\mu}$. We need to determine under which conditions there is spectral stability when $\frac{\rho}{2} > \sqrt{\mu}$. The fact that ω is an integer complicates somewhat the analysis.

Let us suppose that ω belonged to the set of reals. In this case I write $\lambda^{-}(\omega)$. We can find a critical value $\lambda_c^{-} = \rho/2 - \sqrt{\mu}$ that is the maximum for $\lambda^{-}(\omega)$. If $\rho/2 = \sqrt{\mu}$ that critical value will be attained for $\xi_c = \rho/2$, but if $\rho/2 > \sqrt{\mu}$ then there would be two frequencies $\xi^{\pm} = \frac{\rho}{2} \pm \sqrt{\Delta_\omega} = \frac{\rho}{2} - \left(\left(\frac{\rho}{2}\right)^2 + \mu\right)^{\frac{1}{2}}$ such that if $\xi \in (\xi^{-}, \xi^{+})$ then a subset of frequencies exists such that the stable eigenvalue would be positive, i.e $0 < \lambda^{-}(\omega) \leq \lambda_c^{-}$. In this case we can partition the domain of frequencies $\Omega = \mathbb{R}$ into three subsets $\Omega = \Omega^s \cup \Omega^0 \cup \Omega^u$ such that $\lambda^{-}(\omega) > 0$ if $\omega \in \Omega^u$, $\lambda^{-}(\omega) = 0$ if $\omega \in \Omega^0$, and $\lambda^{-}(\omega) < 0$ if $\omega \in \Omega^s$. As $\lambda^{-}(\omega) < 0$ for $\omega \in \{-\infty, 0, +\infty\}$ then $\Omega^0 \cup \Omega^s$ will contain two compact symmetric subsets bounded by finite values for ω, one contained in $\mathbb{R}_{-}$ and the other contained in $\mathbb{R}_{+}$. In any case, if $\rho/2 > \sqrt{\mu}$ the subset Ω^u will always be non-empty.

However, as ω belongs to the set of integers, $\mathbb{Z}$, the conditions for the existence of a subset of frequencies such that the stable eigenfunction is positive are more stringent. Again, let $\rho/2 > \sqrt{\mu}$, and introduce a partition over the set of frequencies: $\mathbb{Z} = \mathbb{Z}^s \cup \mathbb{Z}^0 \cup \mathbb{Z}^u$. In this case, we have again the same critical Fourier modes $\xi^{+} > \xi^{-} \geq 0$, and

$$\mathbb{Z}^u = \left\{\omega \in \mathbb{Z} : -\frac{\sqrt{\xi^{+}}}{2\,\pi\,\tau} < \omega < -\frac{\sqrt{\xi^{-}}}{2\,\pi\,\tau} \text{ or } \frac{\sqrt{\xi^{-}}}{2\,\pi\,\tau} < \omega < \frac{\sqrt{\xi^{+}}}{2\,\pi\,\tau}\right\},$$

such that $\lambda_{-}(\omega) > 0$ for every $\omega \in \mathbb{Z}^u$. As $\lambda_0^{-} < 0$ we can easily see that:

1. If $\tau > \dfrac{\sqrt{\xi}^{+}}{2\pi}$ then $\lambda_\omega^{-} < 0$ for all $\omega \in \mathbb{Z}$, and the set $\mathbb{Z}^u$ is empty.
2. Set $\mathbb{Z}^u$ is non-empty only if $\tau < \dfrac{\sqrt{\xi}^{+}}{2\pi}$.

Summing up, we have two potential distribution dynamics, both satisfying the transversality condition (40), a spectrally stable case when $\lambda_\omega^{-} < 0$ for all $\omega \in \mathbb{Z}$, and a spectrally-unstable case, when there is a non-empty subset $\mathbb{Z}^u$ such that $\lambda_\omega^{-} > 0$ for every $\omega \in \mathbb{Z}^u$ and $\lambda_\omega^{-} < 0$ for all $\omega \in \mathbb{Z}^s$. The existence of spectral stability requires not onlyconditions on the curvature of the utility and the production functions, but also the existence of relatively high barriers to spatial reallocation of capital (that is a small level for τ). [23]

What are the consequences to the distribution dynamics of the existence of spectral instability ?

It can be proven[24] that, under the assumption there is a deviation from the flat steady state, the following occurs.

First, if there is spectral stability then

$$\lim_{t\to\infty} \tilde{K}(t,x) = \lim_{t\to\infty} \mathsf{M}[\tilde{K}](0)e^{\lambda_0^- t} = 0, \text{ for all } x \in \mathrm{X},$$

and

$$\lim_{t\to\infty} \tilde{C}(t,x) = \lim_{t\to\infty} \lambda_0^+ \mathsf{M}[\tilde{K}](0)e^{\lambda_0^- t} = 0, \text{ for all } x \in \mathrm{X},$$

where $\lambda_0^+ = \rho - \lambda_0^- > \rho > 0$. After an arbitrary initial perturbation away from the flat steady state the distribution will converge location-wise to the initial steady state.

Second, if there is spectral instability, there will be two locations $\tilde{x}$ and $-\tilde{x}$, which partition the domain X into two subsets X^+ and X^- (not necessarily compact) such that

$$\lim_{t\to\infty} \tilde{K}(t,x) = \lim_{t\to\infty} \tilde{C}(t,x) = \begin{cases} +\infty & \text{if } x \in \mathrm{X}^+, \\ 0 & \text{if } x \in \{-\tilde{x}, \tilde{x}\}, \\ -\infty & \text{if } x \in \mathrm{X}^-. \end{cases}$$

At last, for both spectrally stable or unstable cases we have

$$\lim_{t\to\infty} \mathsf{M}[\tilde{K}](t) = \lim_{t\to\infty} \mathsf{M}[\tilde{K}](0)e^{\lambda_0^- t} = 0, \text{ for all } x \in \mathrm{X}$$

and

$$\lim_{t\to\infty} \mathsf{M}[\tilde{C}](t) = \lim_{t\to\infty} \lambda_0^+ \mathsf{M}[\tilde{K}](0)e^{\lambda_0^- t} = 0, \text{ for all } x \in \mathrm{X}.$$

[23] However, I should note that we have saddle-point dynamics in a generalized sense because the eigenfunction λ_ω^+ is positive for all $\omega \in \mathbb{Z}$.

[24] See Corollary 3 in the Appendix for the proof.

Therefore, while the profile of the dynamics can tend to a homogeneous distribution or not, the aggregate (or L^1) dynamics always displays stability, in the generalized saddle-point sense,

We can conclude that, following a spatially heterogeneous perturbation from a flat steady state, the location-wise dynamics depends on the spectral dynamic features of the model. If there is spectral stability the previous steady state will be asymptotically restored, for every location. In contrast, if there is spectral instability both capital and consumption distributions will tend to agglomerate in an increasingly reduced subset of locations. However, surprisingly, the dynamics of the aggregate capital and consumption distribution will be the same for both cases: they will behave as in the a-spatial Ramsey model by converging to the initial aggregate steady state levels. Therefore, spectral instability will involve instability in a distributional but not at the aggregate level. When there are decreasing marginal returns, if there is spectral stability we have a stable diffusive dynamics, and if there is spectral instability we have a stable agglomerative dynamics.

4.1.2 An example

To illustrate the previous results, I assume a Cobb-Douglas production function, $Y = f(K) = A\,K^{\alpha}$, where $A > 0$ and $0 < \alpha < 1$, and keep the isoelastic utility function. In this case we have the well known non-singular Ramsey stationary solution,

$$K^* = \left(\frac{\alpha\,A}{\rho+\delta}\right)^{\frac{1}{1-\alpha}},\ C^* = (Q^*)^{-\frac{1}{\theta}} = \psi\,K^*, \tag{48}$$

where $\psi \equiv \dfrac{\rho+(1-\alpha)\delta}{\alpha}$. This implies that μ, in equation (49), takes the parametric form

$$\mu = \mu\big(\alpha,\rho,\delta,\theta\big) \equiv \frac{(1-\alpha)\,(\rho+\delta)\,\psi}{\theta} > 0, \tag{49}$$

the stable eigenfunctions are

$$\lambda_\omega^- = \frac{\rho}{2} - \left[\left(\frac{\rho}{2} - \xi_\omega \right)^2 + \frac{(1-\alpha)(\rho+\delta)\,\psi}{\theta} \right]^{1/2}, \text{ for any } \omega \in \mathbb{Z} \tag{50}$$

and, in particular, the eigenvalue for the frequency $\omega = 0$ is

$$\lambda_0^- = \frac{\rho}{2} - \left[\left(\frac{\rho}{2} \right)^2 + \frac{(1-\alpha)(\rho+\delta)\,\psi}{\theta} \right]^{1/2} < 0.$$

We showed that a necessary condition for the existence of spectral instability is $\rho/2 > \sqrt{\mu}$. We can now have a parametric representation of this condition.[25] Defining

$$\theta^c(\rho) \equiv \frac{4\,(1-\alpha)\,(\rho+\delta)\psi}{\rho^2} \tag{51}$$

we have spectral stability if the elasticity of intertemporal substitution (EIS) is relatively high (if $\theta < \theta^c$), and we have spectral instability only if the EIS is low (only if $\theta > \theta^c$). To have spectral instability the barrier to spatial transfer of capital should also be high, that is τ should have the upper bound $\sqrt{\xi^+}/(2\pi)$.

`Figure 4 around here`

Figure 4 presents a bifurcation diagram in the (α, θ) coordinate space, for given values of ρ and δ. Increasing ρ or δ will move the curve downwards. In that Figure, we have spectral stability for values of the parameters below the line $\theta = \theta^c$, and, possibly spectral instability for values above that line. We see that any of the two types of dynamics can occur for "reasonable" values of the parameters if the production function is close to linear, e.g $\alpha \approx 0.8$. Furthermore, distributional instability is associated to low levels of elasticity of intertemporal substitution $(1/\theta)$ and/or high levels of the rate of time preference.

`Figure 5 around here`

[25] Brito (2004) and Brock and Xepapadeas (2008) address a similar problem by calling it diffusion-induced bifurcation point.

Figure 5 plots the "stable" eigenfunction as a function of frequencies, λ^-_ω, for different values of the EIS parameter, θ, and assuming relatively high barriers to trade associated to a low value for τ. In that figure, the dot lines refers to integer-valued frequencies and the dashed lines refer to real-valued frequencies, and the bottom curve corresponds to a case of spectral stability, in which $\theta < \theta^c$, and the upper curve corresponds to a case of spectral instability, in which $\theta > \theta^c$ and τ is low. The figure represents a particular confirmation of several conclusions from the last subsection: first, the value of the eigenfunction for $\omega = 0$, λ^-_0 is always negative, second, the eigenfunction can only be positive in an open economy environment in which $\tau > 0$ and therefore λ^-_ω is a function of non-zero frequencies; third, the eigenfunction is always negative for high values of ω in absolute value, $\lim_{|\omega|\to\infty} \lambda^-_\omega < 0$; fourth, distributional stability exists if there is spectral stability, that is λ^-_ω is negative for all frequencies as is the case for the bottom curve; and at last, distributional instability is associated to the existence of some frequencies such that the eigenfunction is locally positive.

Figure 5 helps in clarifying the claim that $\theta > \theta^c$ is just a necessary condition, and we need to have a low value for τ. If ω is a real number, the maximum for the stable eigenfunction is λ^c, which is attained for $\omega^c_\mp = \mp\frac{\sqrt{2\rho}}{4\pi\tau}$. The necessary condition $\theta > \theta^c$ implies $\lambda^c > 0$. Furthermore, in the case of the upper curve in that Figure, $|\omega^c|$ it is a real number belonging to the interval $(1, 2)$. This means that there will be at least two integers, ω^0_+ and $\omega^0_- = -\omega^0_+$ such that $\lambda^-_{\omega^0_-} = \lambda^-_{\omega^0_+} > 0$. If we assumed instead a higher value for τ such that, for instance, $|\omega^c| \in (0, 1)$ then, although the necessary condition for instability is satisfied, i.e., $\theta > \theta^c$, all the eigenfunctions λ^-_ω, for $\omega \in \mathbb{Z}$, would be negative, which means we will have spectral stability. A minimal condition to have spectral instability when $\omega \in \mathbb{Z}$ is that $\lambda^-_1 = \lambda^-_{-1} > 0$. Lower values of τ would imply that there are positive eigenvalues for some other (integer valued) frequencies of absolute values higher than one.

Figures 6 and 7 illustrate the dynamic adjustment of capital stock, consumption and net savings for the distributional stable and unstable cases. The left panels show the ensemble distribution, and right panels show the aggregates. To draw those figures it is assumed that there is an initial distribution of the capital stock away from a normalized steady state, such that $K^* = 1$, given by the distribution

$$k_0(x) = \beta \frac{e^{-2\beta|x|}}{1 - e^{-\beta}}, \; x \in \mathrm{X}$$

for $\beta > 0$. This distribution satisfies $\mathsf{M}[K](0) = \int_{-1/2}^{1/2} k_0(x)dx = 1$ which implies that the initial variation of the capital stock satisfies $\tilde{K}\big(0, -\frac{1}{2}\big) = \tilde{K}\big(0, \frac{1}{2}\big) = 0$. This means that the center locations have a higher increase and the peripheral locations has a lower, or even zero, increase in their capital stocks.

They provide a numerical confirmation to our previous qualitative results. Let us start by looking at the left-hand side diagrams, depicting the dynamics for the distributions profile, first for the spectrally stable case, in Figure 6, and next to the spectrally unstable case, in Figure 7.

At time $t = 0$ the spatially heterogeneous deviation of the initial capital distribution from a flat steady state (see Subfigure 6a) causes an increase in output that will be stronger in the central than in the peripheral locations. On impact consumption (see Subfigure 6b) also increases across all locations, but more importantly in the central locations rather than in the peripheral locations. The increase in consumption is generally high than the increase in output, which implies negative savings across all locations, but, in particular in the central locations (see Subfigure 6c).

The effects that produce this response of savings are contradictory: one hand, the fact that the initial distribution is locally convex will generate a spatial flow of capital from the periphery to the center, but, on the other hand, the initial shift in the capital distribution will reduce the marginal return of capital in the central locations relative to the peripheral locations, which would generate incentives for a spatial flow of capital in the opposite direction. Looking at the diagrams it seems that the second effect is dominant, which implies a outflow of capital from the central locations to the peripheral locations. Whatever the dominant direction of the spatial capital flows, that depend on the joint effects of the friction in the transportation of capital and of the spatial wedge in the value of capital, the dynamics of savings for the central locations is dominated by the intertemporal effect of the reduction in the marginal return of capital in the intertemporal allocation of consumption, and therefore on savings.

Over time, the negative savings in central locations will drive down local capital accumulation. This process will increase the rate of return of capital in the central locations, and will reduce its spatial heterogeneity. This will progressively reduce the incentive for dissaving in the central locations and reduce the incentives relocating capital across locations. The economy will converge to the original steady state in a relatively fast transition process.

Looking at Figure 7 for the spectrally unstable case, we see that the initial impact on capital is the same as for the spectrally stable case (see Subfigure 7a). But although the variation on impact, at time $t = 0$, for consumption and savings are qualitative the same, for ensemble distribution, they are very different in their distributional profiles (see Subfigures 7b and 7c): there is also a general increase in consumption, but it is now almost homogeneous across space; and there are also negative savings, but it is now much bigger in the peripheral locations than in the central locations. This different variation in savings across space generates also a different spatial dynamics.

The previous effects that were identified, to explain the behavior of savings, still operate when there is spectral instability but the relative strength is different. The explanation for the initial response of savings in Subfigure 7c is that the effect of the convexity in the initial shock to the capital distribution dominates both the effect of the regional differences in the rate of return of capital, and of the fall in the rate of return of capital on the intertemporal substitution of consumption. The resultant is that there is a spatial reallocation of capital in the opposite direction, from the periphery to the center, and the incentive for reducing the local accumulation of the capital stock in the central locations is much smaller. This can be explained, on one hand, by the existence of higher barriers to spatial flows of capital and by a reversal in the effect of the spatial wedge in the regional allocation of savings, and, on the other hand, by the existence of a smaller intertemporal elasticity of substitution, which reduces the response of savings to the local reduction in the marginal return of capital in the center.

Over time there is a progressive concentration of capital in central locations coming from peripheral locations. The process of agglomeration dominates an underlying process of diffusion, that is still operative because the ensemble distribution of capital, as can be seen in Subfigure 7a, will became locally concave in central locations. Convergence to an asymptotic bounded distribution is governed by the effect of both the deceasing returns to capital and a weak process of redistribution of capital away from central to peripheral locations.

At last, turning our attention now to the dynamics of the aggregates, which are illustrated in the right-hand side diagrams of Figures 6 and 7, we see that the dynamics are qualitatively similar for both the spectral stable and unstable cases. There is a convergence of the aggregate capital stock, consumption, and savings to the initial aggregate steady state. There is stability in the L^1 sense. Therefore, spectral instability does not mean we have instability in the mean but instead there is asymptotically heterogeneity in the distribution, meaning that the diffusive character of the stable spectral dynamics, temporarily countered by the unstable spectral dynamics, which has an agglomerative character. Therefore, the process of capital accumulation is ergodic in the distributional Ramsey case, and it is non-ergodic in the distributional AK case.

`Figure 6 around here`

`Figure 7 around here`

4.1.3 The role of spectral dynamics

Comparing the behavior of the analogous spectral growth factor λ^-_ω for the decreasing returns case (DRS), in equation (46) and in Figure 5, for the distributional instability case, with the analog for the constant returns to scale case (CRS), in equation (25) and Figure 1, clarifies both the crucial effect on dynamics of that function, and the main differences between the the distributional Ramsey and AK models. First, the value of the eigenfunction for $\omega = 0$ drives the aggregate of the ensemble distribution, it is positive for the CRS case and negative for both DRS cases, which leads to long-run growth in the aggregate in the all the versions of the AK model, and to the absence of long-run growth in the aggregate in all the versions of the Ramsey model. Second, the effect of the spatial reallocation is potentially different for the two types of technologies. While frequencies greater than zero have a diffusive nature in the AK case, they have a potential agglomerative effect in the Ramsey case.

The pattern-formation literature studies the dynamics of two-dimensional forward parabolic PDEs from a spectral stability perspective. In this literature, a bifurcation scenario in which $\omega_{\pm}^{c} \neq 0$ and $\mathrm{Im}(\lambda^*(\xi^c, \theta^c)) = 0$ is called a I_s instability case (see Cross and Hohenberg (1993)), which is a bifurcation that unfolds a type of instability associated to the formation of time-stationary spatial patterns. This type of instability is called Turing instability (see Turing (1952)), and can occur when the local approximation of a two-dimensional systems of forward parabolic PDEs has two eigenvalues with negative real parts. Spatial pattern-formation may be a mechanism that generates distributional stability when there is spectral instability. Turing instability has also been found to exist in the context of spatially heterogeneous models as a mechanism leading to the emergence of agglomeration.[26] However, in our model we have instead a two-dimensional system forward-backward parabolic PDEs, representing the first order conditions of an optimal distributed control problem.[27] This means that although there is spectral instability it is not of the previous pattern-formation type. Furthermore, the case in which there is convergence to a distributional BGP with positive growth rate is only possible if there is non-stationarity in a distributional sense, which does not exist in two-dimensional forward PDEs.

4.2 Comparative dynamics for a productivity shock

Assuming that the economy is at a homogenous steady state (K^*, C^*), for every $x \in \mathrm{X}$, assuming a given total factor productivity level $A = A_0$. In this section I study the effects

[26](Fujita et al., 1999, ch. 6) proved that spatial interaction among symmetric locations may eventually lead to the emergence of agglomeration, through a mechanism of pattern formation.

[27]As in Brito (2004) and Brock and Xepapadeas (2008).

of a spatially heterogeneous productivity shock from A_0 to $A_1(x)$. Therefore, we have the spatial variation $\tilde{A}(x) = A_1(x) - A_0$. I assume there is spectral stability.[28]

The production function is rewritten as $f(A, k)$, and the derivatives, evaluated at the steady state, are denoted now by $f_A(K^*)$, $f_K(K^*)$, $f_{AK}(K^*)$ and $f_{KK}(K^*)$. Recall that at the steady state $f_K(K^*) = \rho + \delta$

Proposition 5. *Consider an economy in which the local dynamics from a flat steady state is represented by PDE system (36)-(37), and assume that it is distributionaly stable. Then, the short-run multipliers for a non-anticipated, permanent, and spatially-heterogeneous productivity variation $\tilde{A}(x)$ are*

$$\tilde{K}(t,x) = \bar{K}(x) - \int_{-1/2}^{1/2} \tilde{A}(y)\, m^k(t, x-y)\, dy, \text{ for each } (t,x) \in \mathrm{T} \times \mathrm{X} \tag{52}$$

$$\tilde{Q}(t,x) = \bar{Q}(x) - \int_{-1/2}^{1/2} \tilde{A}(y)\, m^q(t, x-y)\, dy, \text{ for each } (t,x) \in \mathrm{T} \times \mathrm{X} \tag{53}$$

where the long-run multipliers are

$$\bar{K}(x) = \int_{-1/2}^{1/2} \tilde{A}(y)\, \bar{m}^k(x-y)\, dy, \text{ for each } x \in \mathrm{X} \tag{54}$$

$$\bar{Q}(x) = \int_{-1/2}^{1/2} \tilde{A}(y)\, \bar{m}^q(x-y)\, dy, \text{ for each } x \in \mathrm{X} \tag{55}$$

where, $\bar{m}^k(x) = \sum_{\omega\in\mathbb{Z}} \mathcal{M}^k_\omega\, e^{i2\pi\omega x}$ *and* $\bar{m}^q(x) = \sum_{\omega\in\mathbb{Z}} \mathcal{M}^q_\omega\, e^{i2\pi\omega x}$, *and*

$$m^k(t,x) = \sum_{\omega\in\mathbb{Z}} \mathcal{M}^k_\omega\, e^{i2\pi\omega x + \lambda^-_\omega t}, \text{ and } m^q(t,x) = \sum_{\omega\in\mathbb{Z}} \psi^-_\omega \mathcal{M}^k_\omega\, e^{i2\pi\omega x + \lambda^-_\omega t},$$

[28] Our approach is qualitative. However, the existence spectral instability may not be admissible from the economic point of view, because solutions for the capital stock and consumption should be positive, at every point in space. This is only possible to prove if I could find closed form solutions to problem, which I guess cannot be found. This is the reason why I restrain this section to the spectrally stable case. We can see this as an application of Samuelson's correspondence principle.

where λ_ω^- *is in equation* (46), $\psi_\omega^- = u''(C^*)\,(\lambda_\omega^+ - \xi_\omega)$, *and*

$$\mathcal{M}_\omega^k \equiv -\frac{\xi_\omega\, f_A(K^*) - \frac{Q^*}{u''(C^*)}\, f_{AK}(K^*)}{\xi_\omega\,(\rho - \xi_\omega) - \mu}, \;\; \textit{and}\; \mathcal{M}_\omega^q \equiv -\frac{Q^*\left(f_A(K^*)\, f_{KK}(K*) + (\xi_\omega - \rho) f_{AK}(K^*)\right)}{\xi_\omega\,(\rho - \xi_\omega) - \mu}.$$

Assuming distributional stability, the long-run multipliers for the level and the marginal value of capital are, in general, spatially heterogeneous, [29]

$$\lim_{t\to\infty} \tilde{K}(t,x) = \bar{K}(x), \text{ and } \lim_{t\to\infty} \tilde{Q}(t,x) = \bar{Q}(x), \text{for } x \in \mathrm{X}.$$

In order to illustrate this result, I consider again the benchmark case, with an isoelastic utility function, and Cobb-Douglas production function, with yields

$$\begin{aligned}\mathcal{M}_\omega^k &= -(\rho+\delta)\left(\frac{\xi_\omega}{\alpha} + \frac{\psi}{\theta}\right)\frac{K^*}{\xi_\omega\,(\rho - \xi_\omega) - \mu}\\ \mathcal{M}_\omega^q &= (\rho+\delta)\left(\frac{\xi_\omega - \psi}{\theta}\right)\frac{C^*}{\xi_\omega\,(\rho - \xi_\omega) - \mu}.\end{aligned}$$

In addition, I consider an heterogeneous proportional productivity shock

$$\tilde{A}(x) = A_0\left(e^{\beta\left(\frac{1}{2} - |x|\right)} - 1\right), \text{ for } x \in \mathrm{X},$$

where $\beta > 0$. This implies that $\tilde{A}(x)$ is non-negative, has a positive maximum for $x = 0$, and satisfies $\tilde{A}\big(\frac{1}{2}\big) = \tilde{A}\big(-\frac{1}{2}\big) = 0$. Again, the TFP shock is bigger for central locations and smaller, eventually zero, for peripheral locations.

Applying these formulas to the expressions for the multipliers for K and Q in Proposition 5, we can obtain the time- and location-dependent multipliers for consumption, $\tilde{C}(t,x) = \tilde{Q}(t,x)/u''(C^*)$, for output, $\tilde{Y}(t,x) = K^*\,\tilde{A}(x) + (\rho+\delta)\,\tilde{K}(t,x)$, and savings, $\tilde{S}(t,x) =$

[29] Although the model in this section does not have a closed form solution, this result provides us on an indication of the type of long-run distribution one would expect if there would be heterogeneity in the TFP across locations and not homogeneity as I have assumed in this section. For this when there is constant returns to scale see Boucekkine et al. (2019b).

$\tilde{Y}(t,x) - \tilde{C}(t,x)$. Figure 8 illustrates the behavior of those multipliers.

`Figure 8 around here`

At time $t = 0$, differently from the case in section 4.1.1, capital does not change. It will only change over time through investment (see Subfigure 8a). Immediately after the shock there is an increase (on impact) in output and in consumption in all locations, see Subfigures 8b and 8c. The increase in output is a direct consequence of the TFP shock, because the capital stock is still at the level of the initial steady state. The increase in consumption takes place in all locations, although it is larger for central locations, as results from the anticipation of future increases in income. This implies that the change in savings is also heterogenous, but more importantly, it changes in a different direction across space: it is positive for central locations and negative for peripheral locations.

Over time, that change in savings generates a process of capital accumulation in the central locations coming both from home savings and from capital reallocation from peripheral locations, see Subfigures 8a and 8d. Although, there is a reduction in capital stock in peripheral locations, consumption never decreases. This can only happen if there is consumption reallocation towards peripheral regions over time.

Both processes, of capital reallocation from peripheral to central locations and of consumption in the opposite direction, generate permanent positive savings in central locations and negative savings in peripheral locations. Asymptotically, although we started from a flat steady state, there will be heterogeneity in the distributions of capital, income and consumption: there are long-run increase in the stock of capital, income and consumption in central locations, but peripheral locations, although increasing their consumption, will have a lower level of income and capital (see right-hand-side figures in Figure 8). However, the aggregate levels of all those variables will increase in the long run.

There are two effects that command the incentives for the intertemporal and inter-spatial allocation of capital: a direct effect resulting from an increase in productivity, which is higher for the central locations, and the indirect effects related to the change in the capital allocation both through time and space that I referred in section 4.1.1. In particular, we saw that, when there is spectral stability, the effect of the change in the marginal productivity of capital dominates both the reallocation of capital across space (flowing from the central to the peripheral locations) and over time (the accumulation of capital reduces the incentive for savings over time). In subsection 4.1.1 the indirect effects were the only ones that existed. In this comparative dynamics exercise, the spatial capital reallocation over time, from peripheral to central locations, tells us that the first direct effect dominates all the other indirect effects in the determination of savings over time along the adjustment process. However, the indirect mechanisms eventually prevail in forcing the evolution of distributions to converge towards stationarity.

Savings has an apparently puzzling adjustment: on one hand, central locations display an increase in income higher than the increase in consumption, which generate long-run positive savings, while peripheral locations have the opposite development, with asymptotic negative savings. But, on the other hand, aggregate savings are positive but decrease over time tending to zero.

This should be understood in the context of an open economy in which there is spatial reallocation of capital and savings even in the steady state. There is a stationary long-run distribution of capital if the allocation of capital stock per location does not change. This is only possible if there is a redistribution of capital across space that exactly matches the differences in TFP across locations (recall that, in the case studied in section 4.1.1, the TFP was homogeneous). When there is heterogeneity in TFP the asymptotic distribution satisfies, in the Cobb-Douglas case

$$\tau^2 \frac{\partial^2 \bar{K}(x)}{\partial x^2} + \bar{S}(x) + \delta\, \bar{K}(x) = 0, \text{ for each } x \in \mathrm{X}, \tag{56}$$

where $\bar{S}(x) = A(x)\bar{K}(x)^{\alpha} - \bar{C}(x)$. Therefore, if $A(x)$ is not flat, then both $\bar{S}(x)$ and $\bar{K}(x)$ will not be flat as well. But as

$$\bar{S}(x) \gtreqless 0, \iff \tau^2 \frac{\partial^2 \bar{K}(x)}{\partial x^2} + \delta\, \bar{K}(x) \lesseqgtr 0, \text{ for any } x \in \mathrm{X}.$$

then, for locations with positive savings, the ensemble distribution of capital is locally concave. Again this is only possible if the direct effect on incentives of the increase in productivity, which generates inward flows, dominates the incentive effect, which, with our assumption on the direction of capital flows, should generate asymptotically outward flows of capital for central locations.

Equation (56) is essentially a static balance of payments equilibrium condition, in the absence of net capital accumulation (out of depreciation). Therefore at the ensemble economy level, as can be seen Subfigure 8d, the aggregate savings is

$$\mathsf{M}[\bar{S}] = \int_{-1/2}^{1/2} \bar{S}(x)dx = 0$$

and therefore,

$$\int_{-1/2}^{1/2} \tau^2 \left(\frac{\partial^2 \bar{K}(x)}{\partial x^2} + \delta\, \bar{K}(x)\right) dx = 0.$$

5 Conclusion

Introducing spatial heterogeneity in the benchmark AK-Ramsey model, allows for dealing with distributional dynamics converging over time to a balanced growth path or to a steady state, depending on the technology having constant or decreasing returns to scale. In this paper we clarify a difference between convergence at the aggregate (or L^1) level, and at the distribution (or location-wise) level, and the mechanisms leading to diffusion or agglomeration of capital across space. We show that conducting a bifurcation analysis on the spectra of frequencies, coming from a Fourier transformation to the solution of the model allows for the identification of the characteristics of the distribution dynamics.

We show that the same forces leading to long run growth, in the a-spatial AK model, or to saddle-point dynamics converging to a steady state, in the a-spatial Ramsey model, govern the ensemble distribution (in the L^1 sense). However, heterogeneity in space drives the distributional characteristics of the location-wise convergence over time. In a nutshell, we found that the spatial allocation of capital is driven by the size of the barriers for spatial allocation of capital, and by the local elasticity of intertemporal substitution (EIS): low barriers and high EIS imply a dominance of a diffusion mechanism, and high barriers and low EIS imply the dominance of an aggregative mechanism, on the distribution dynamics of capital accumulation.

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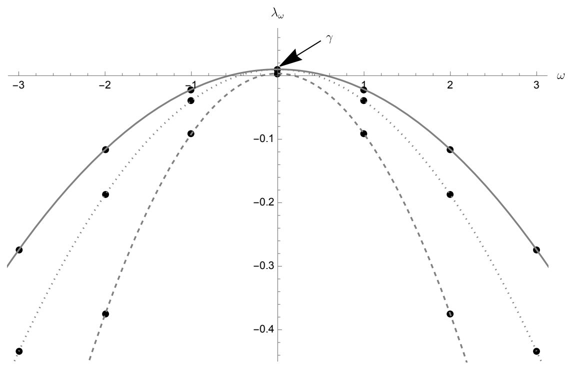
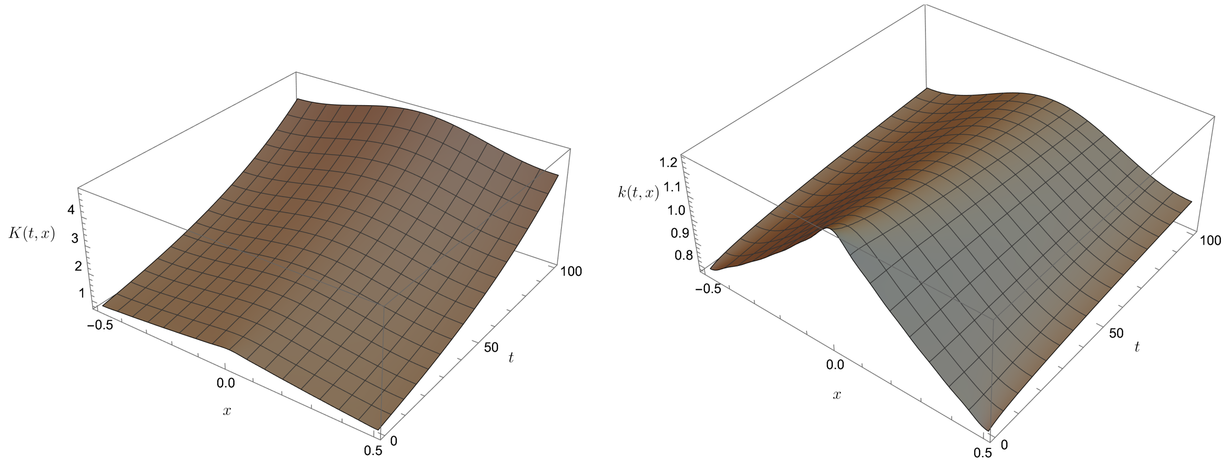
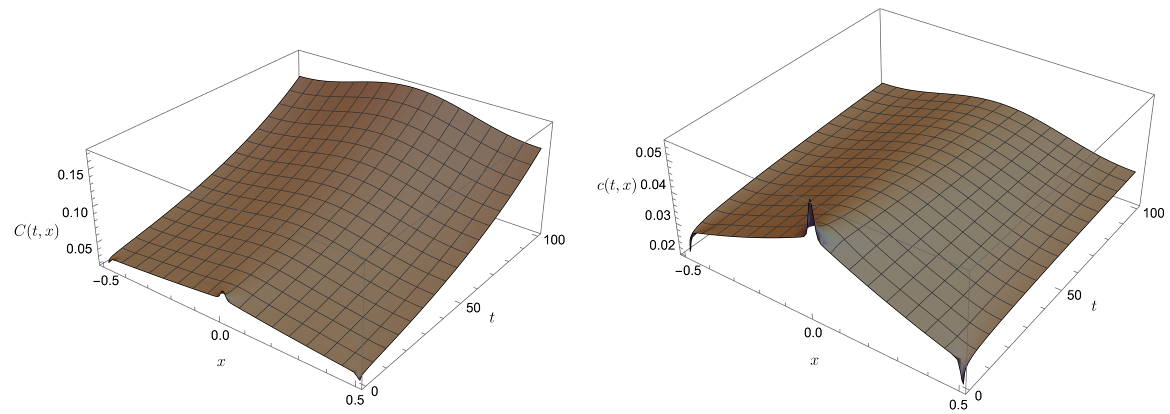


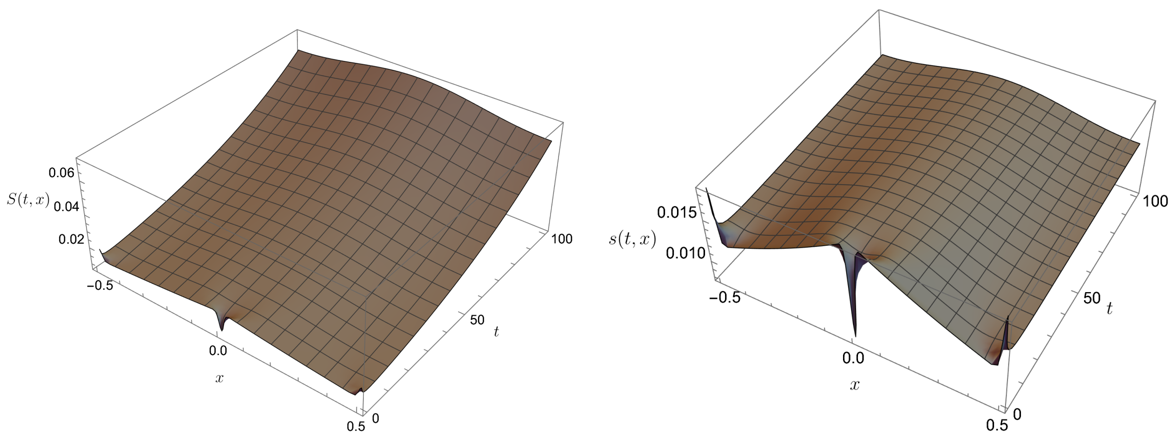
Figure 1: Eigenfunction $\lambda(\omega)$: the dots represent it as a function of $\omega \in \mathbb{Z}$, and the curves represent it as a function of $\omega \in \mathbb{R}$, for different values of the parameters. It is assumed that $\rho = 0.02$, $r = 0.04$, solid curve is for $\tau = 0.02$ and $\theta = 2$, the dotted line is for $\tau = 0.025$ and $\theta = 2$, and the dashed curve for $\tau = 0.02$ and $\theta = 6$.



(a) Capital stock: level and detrended solution



(b) Consumption: level and detrended solution



(c) Savings: level and detrended solution

Figure 2: Dynamics for the CRS case with parameter values $\rho = 0.03$, $\delta = 0.05$, $\theta = 2$, $A = (\theta + \delta)/\alpha$, $\beta = 0.5$ and $\tau = 0.02$. Detrended variables are multiplied by the exponential factor $e^{-\gamma t}$.

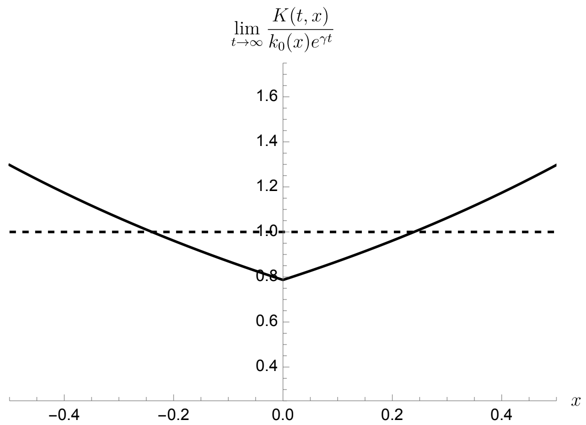


Figure 3: Limit of asymptotic detrended capital distribution versus initial distribution

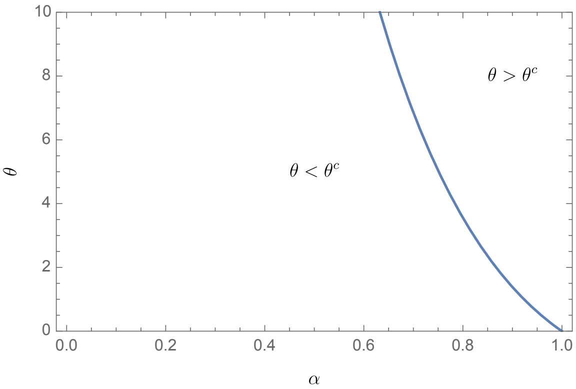


Figure 4: Bifurcation diagrams in the (α, θ) space for $\delta = 0.05$ and $\rho = 0.02$. If $\theta < \theta^c$ there is spectral stability, and $\theta > \theta^c$ is a necessary condition for spectral instability.

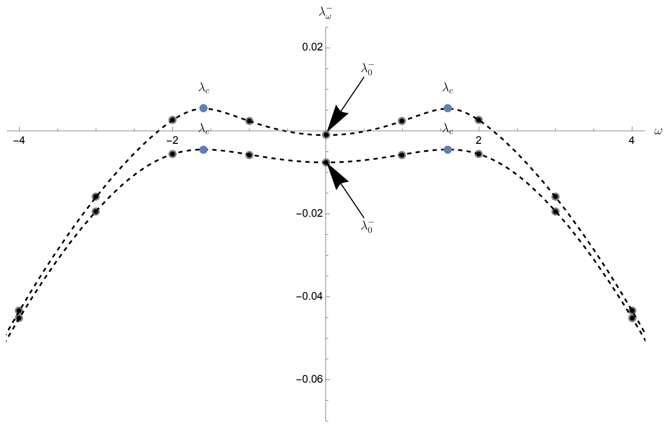
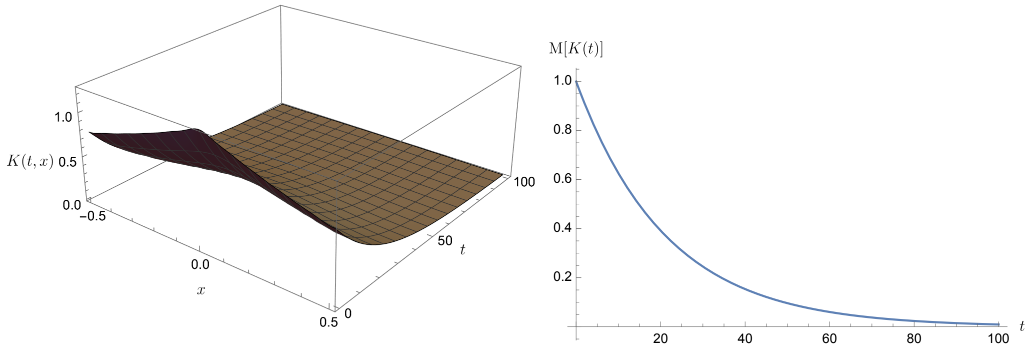
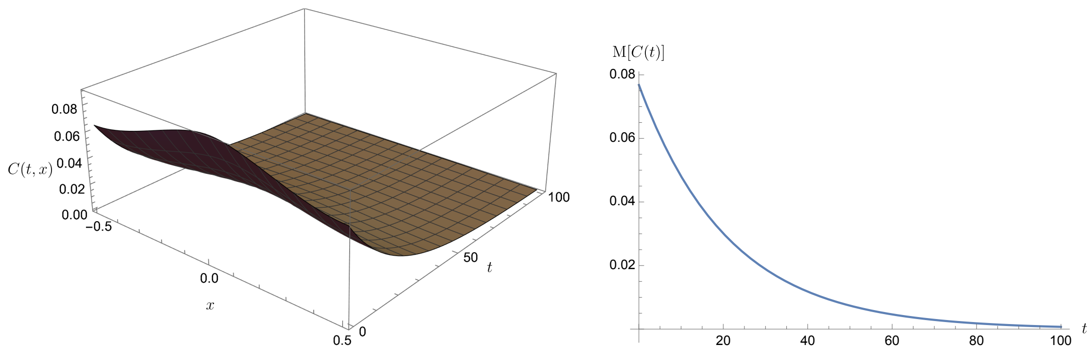


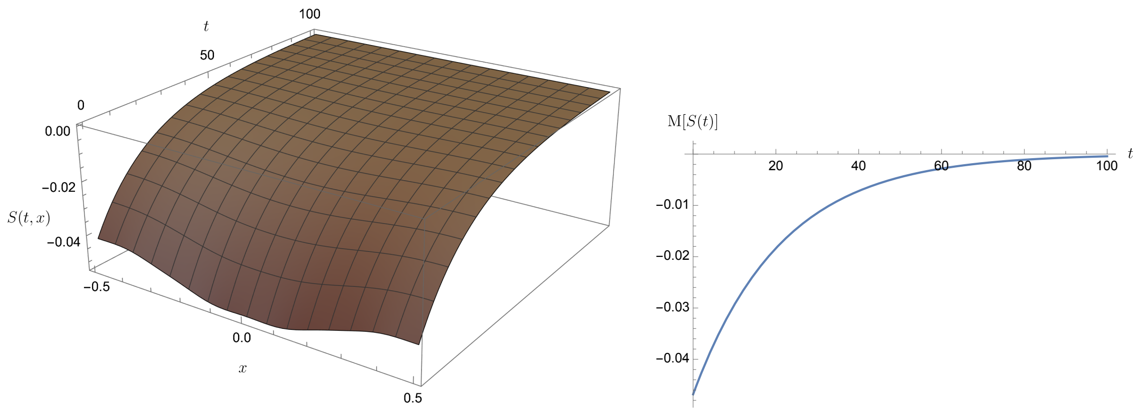
Figure 5: The eigenfunction λ_ω^- as a function of ω for $\rho = 0.02$, $\delta = 0.05$ and $\tau = 0.01$. In this figure, we set $\theta = 2$ for the lower curve and $\theta = 9$ for the upper curve. We also indicate the values for λ_0^- which are always negative.The dot lines represent the case in which ω are integers and the dashed lines represent the case in which ω are real numbers.



(a) Capital stock dynamics: distribution and aggregate

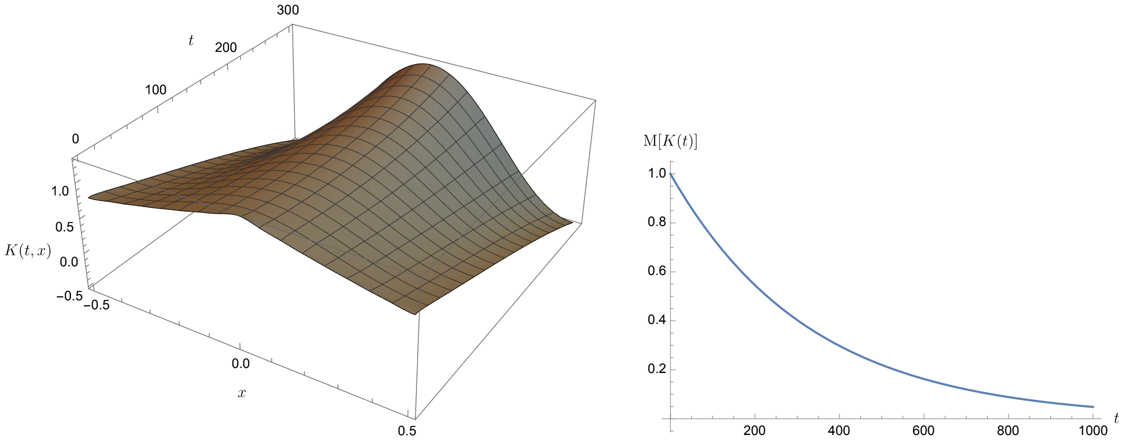


(b) Consumption dynamics: distribution and aggregate

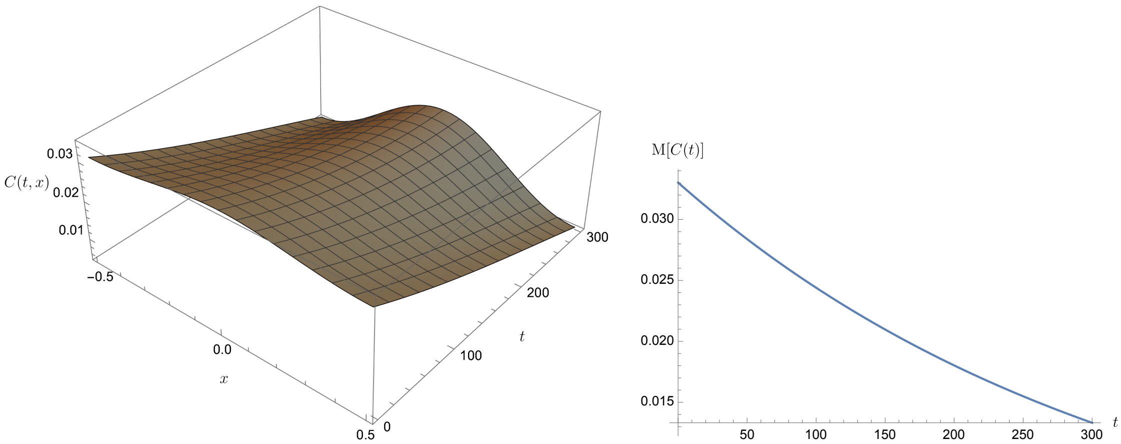


(c) Savings dynamics: distribution and aggregate

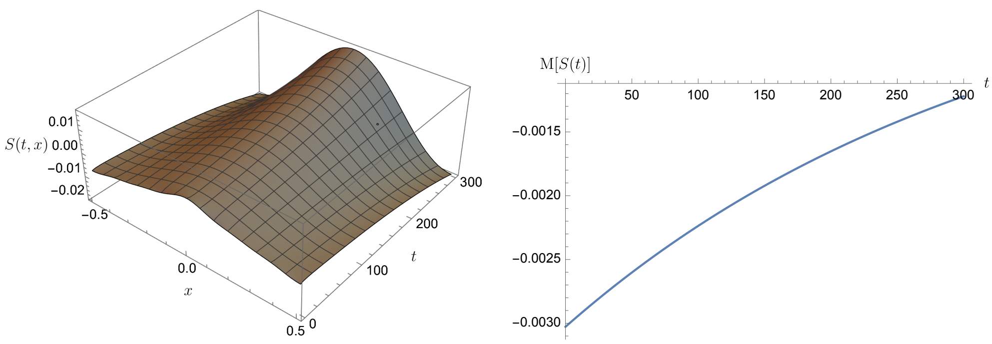
Figure 6: Dynamics for the stable case: $\rho = 0.03$, $\delta = 0.05$, $\alpha = 0.4$, $\theta = 2$, $A = (\theta + \delta)/\alpha$, $\beta = 0.5$, $\tau = 0.02$ and $\mathrm{M}[\tilde{K}(0)] = 1$



(a) Capital stock dynamics: distribution and aggregate

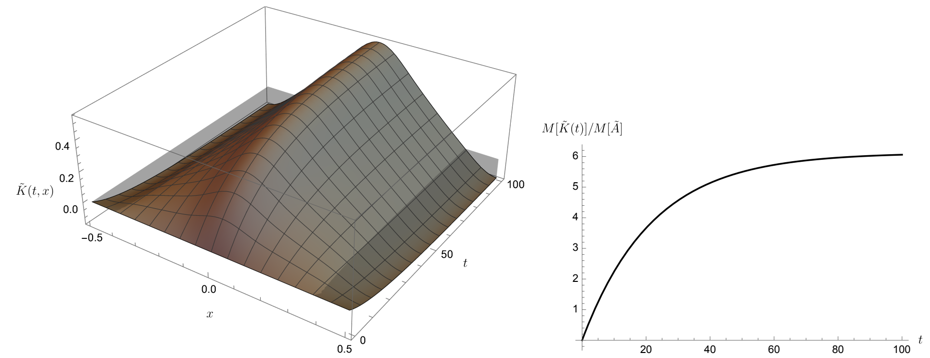


(b) Consumption dynamics: distribution and aggregate

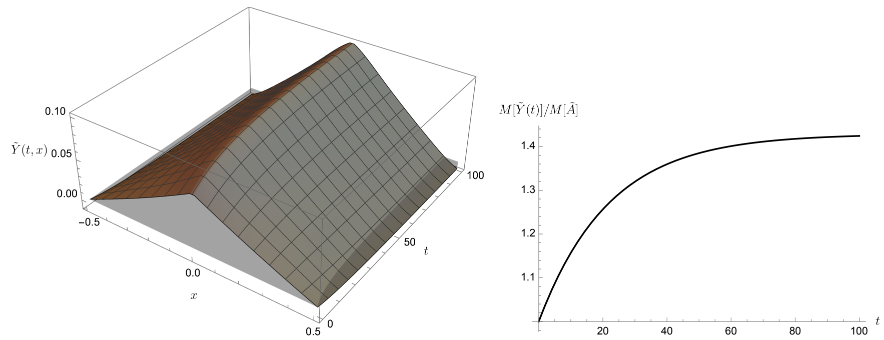


(c) Savings dynamics: distribution and aggregate

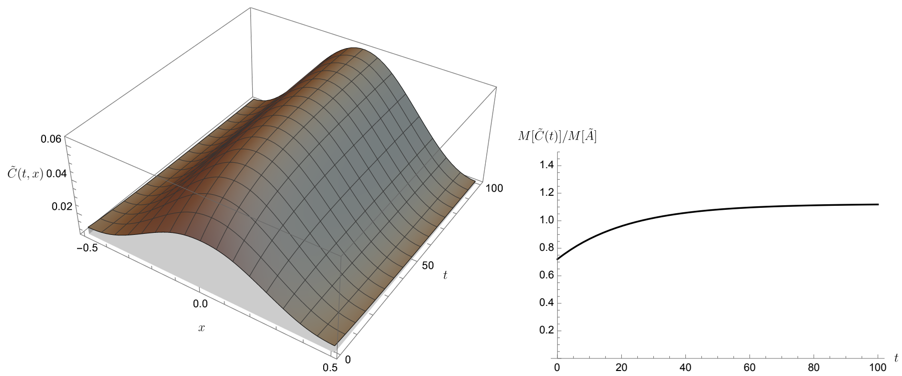
Figure 7: Dynamics for the unstable case: $\rho = 0.03$, $\delta = 0.05$, $\alpha = 0.8$, $\theta = 8$, $A = (\theta + \delta)/\alpha$, $\beta = 0.5$, $\tau = 0.02$ and $\mathrm{M}[\tilde{K}(0)] = 1$



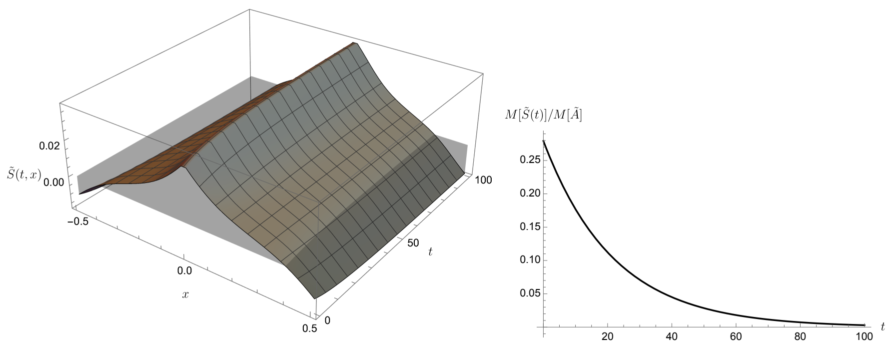
(a) Capital stock multiplier: distribution and aggregate



(b) Output multiplier: distribution and aggregate



(c) Consumption multiplier: distribution and aggregate



(d) savings multiplier: distribution and aggregate

Figure 8: Multipliers for a shock in A

Appendix: not for publication

Lemma 1. *The social welfare functional (7) satisfies the Pigou-Dalton transfer principle.*

Proof of Lemma 1. Consider the functional (7) for a given distribution flow $\big(C(t,x)\big)_{(t,x)\in \mathrm{T}\times\mathrm{X}}$. Now consider a perturbation from $C(t,x)$ to $\tilde{C}(t,x) = C(t,x) + \eta(t,x)$, where $\eta : \mathrm{T}\times\mathrm{X} \to \mathbb{R}_{++}$. The variation in welfare is given by the Fréchet derivative

$$\nabla_C \mathsf{W}[C] = \frac{1}{L}\int_{-L/2}^{L/2}\int_0^{\infty} u'\big(C(t,x)\big)\,\eta(t,x)\,e^{-\rho t}\,dt\,dx.$$

Assume that there are two locations x and x', where $x' = x + \Delta x$, for $\Delta x > 0$, such that $C(t,x') \geq C(t,x)$ at time t. Introduce two "spike" perturbations at the two locations, x and x', such that for any $(s,\xi) \in \mathrm{T}\times\mathrm{X}$ we have

$$\tilde{C}(s,\xi) = C(s,\xi) + \eta(s,\xi),\ \text{where}\ \eta(s,\xi) = \begin{cases} \Delta c\,\delta(s-t,\xi-x) \\ -\Delta c\,\delta(s-t,\xi-x'), \end{cases}$$

where Δc is a positive constant. This means that, at time t, there is a transfer of consumption from the relatively rich location x' to the relatively poor location x.

Using the previous definition of the marginal variation in welfare we find

$$\nabla_C \mathsf{W}[C] = \Big(u'\big(C(t,x)\big) - u'\big(C(t,x')\big)\Big)\,\Delta c\,e^{-\rho t} \geq -u''\big(C(t,x')\big)\,\Delta c\,e^{-\rho t} \geq 0$$

as $C(t,x') \geq C(t,x)$. This means that transferring consumption from locations with higher consumption to locations with lower consumption increases social welfare, which is consistent with the Pigou-Dalton transfer principle. □

Proposition 1. *Assume an optimal allocation $\big(C^*(t,x), K^*(t,x)\big)_{(t,x)\in\mathrm{T}\times\mathrm{X}}$, satisfying the admissibility conditions (4), (5), (6), and (8), exists. Then there is a positively-valued distributional co-state variable $Q : \mathrm{T}\times\mathrm{X} \to \mathbb{R}_{++}$, which is a piecewise continuous function of time, such that the following conditions are satisfied:*

1. *the static optimality condition*

$$u'(C^*(t,x)) = Q(t,x), \text{ for each } (t,x) \in \mathrm{T} \times \mathrm{X} \tag{1}$$

2. *the generalized Euler equation*

$$\frac{\partial Q(t,x)}{\partial t} = -\tau^2 \frac{\partial^2 Q(t,x)}{\partial x^2} + Q(t,x)\Big(\rho - r(K^*(t,x))\Big), \text{ for each } (t,x) \in \mathrm{T} \times \mathrm{X} \tag{2}$$

   where $r(K) = f'(K) - \delta$ *is the marginal productivity of capital net of depreciation;*

3. *the dual boundary conditions*

$$Q\big(t, -\tfrac{L}{2}\big) = Q\big(t, \tfrac{L}{2}\big), \text{ and } \frac{\partial Q\big(t,-\frac{L}{2}\big)}{\partial x} = \frac{\partial Q\big(t,\frac{L}{2}\big)}{\partial x}, \text{ for each } t \in \mathrm{T}, \tag{3}$$

4. *and the transversality condition*

$$\lim_{t\to\infty} Q(t,x)\, K^*(t,x)\, e^{-\rho t} = 0, \text{ for each } x \in \mathrm{X}. \tag{4}$$

Proof of Proposition 1. We have an aggregate optimal control problem of a quasi-linear parabolic partial differential equation. Necessary first-order conditions can be determined by the application of the Ekeland (1974) variational principle. I adopt a heuristic approach. [30] Assuming we know a solution for the problem, (C^*, K^*), the value functional takes the value

$$\mathsf{W}[C^*, K^*] = \frac{1}{L}\int_{-L/2}^{L/2}\int_0^{\infty} u\big(C^*(t,x)\big)\, e^{-\rho t}\, dt\, dx.$$

[30] The literature provides us with several derivations of the optimality conditions, with different levels of generality. I adopt a heuristic approach following, as in Derzko et al. (1984), (Neittaanmaki and Tiba, 1994, ch. 4), Raymond and Zidani (1999), and, in particular Gel'fand and Fomin (1963). See (Li and Yong, 1995, p. 162), (Fattorini, 1999, sec. 11.5) and (Tröltzsch, 2010, ch. 5) for more general functional analytic applications of the Ekeland variational principle to distributed control problems. Bensoussan (1988) provides necessary conditions for an aggregate optimal control case similar to ours.

Consider a small continuous admissible perturbation of the solution $\big(C^*(t,x), K^*(t,x)\big)_{(t,x)\in \mathrm{T}\times\mathrm{X}}$ along the direction $(h_c(t,x), h_k(t,x))$, such that the perturbed solution is $\big(C(t,x), K(t,x))\big)_{(t,x)\in \mathrm{T}\times\mathrm{X}}$ for $C(t,x) = C^*(t,x) + \epsilon\, h_c(t,x)$ and $K(t,x) = K^*(t,x) + \epsilon\, h_k(t,x)$, where ϵ is an arbitrary small positive constant. Any admissible perturbation should satisfy the following conditions: $h_k\big(t,-\frac{L}{2}\big) = h_k\big(t,\frac{L}{2}\big)$ and $\frac{\partial h_k\left(t,-\frac{L}{2}\right)}{\partial x} = \frac{\partial h_k\left(t,\frac{L}{2}\right)}{\partial x}$ for every $t \in \mathrm{T}$ and $h_k(0,x) = 0$, for every $x \in \mathrm{X}$.

The value for any admissible strategy is

$$\mathsf{W}[C,K] = \frac{1}{L}\int_{-L/2}^{L/2}\int_0^\infty u\big(C(t,x)\big)\, e^{-\rho t}\, dt\, dx.$$

Introducing the (current value) co-state variable $Q(t,x)$, and a Lagrange multiplier associated with the terminal condition (8), $\mu(t,x)$, we have

$$\begin{aligned}\mathsf{W}[C,K] = \frac{1}{L}\int_{-L/2}^{L/2}\int_0^\infty e^{-\rho t}\Bigg\{u\big(C(t,x)\big) - Q(t,x)\Bigg[\frac{\partial K(t,x)}{\partial t} - \tau^2\,\frac{\partial^2 K(t,x)}{\partial x^2} - f\big(K(t,x)\big) + \\ + C(t,x) - \delta\, K(t,x)\Bigg]\Bigg\}\, dt\, dx + \lim_{t\to\infty}\frac{1}{L}\int_{-L/2}^{L/2} e^{-\rho t}\,\mu(t,x)\, K(t,x)\, dx,\end{aligned}$$

where the Kuhn-Tucker condition $\lim_{t\to\infty}\mu(t,x)\, e^{-\rho t} K(t,x)\, dx = 0$ holds, for every $x \in \mathrm{X}$. Integrating by parts yields

$$\begin{aligned}\int_0^\infty e^{-\rho t} Q(t,x)\,\frac{\partial K(t,x)}{\partial t} dt &= \int_{t=0}^{t=\infty} e^{-\rho t} Q(t,x)\, K(t,x) \\ &\quad - \int_0^\infty e^{-\rho t}\left(\frac{\partial Q(t,x)}{\partial t} - \rho\, Q(t,x)\right) K(t,x)\, dt \\ \int_{-L/2}^{L/2} Q(t,x)\,\frac{\partial^2 K(t,x)}{\partial x^2} dx &= \int_{-L/2}^{L/2}\left(Q(t,x)\,\frac{\partial K(t,x)}{\partial x} - \frac{\partial Q(t,x)}{\partial x}\right) K(t,x) \\ &\quad + \int_{-L/2}^{L/2}\frac{\partial^2 Q(t,x)}{\partial x^2}\, K(t,x)\, dx.\end{aligned}$$

Therefore

$$
\begin{aligned}
\mathsf{W}[C,K] = {} & \frac{1}{L}\int_{-L/2}^{L/2}\int_0^{\infty} e^{-\rho t}\left\{ u\big(C(t,x)\big) + \left[\frac{\partial Q(t,x)}{\partial t} + \tau^2\,\frac{\partial^2 Q(t,x)}{\partial x^2} - \rho\, Q(t,x)\right] K(t,x) \right. \\
& \left. - Q(t,x)\left[f\big(K(t,x)\big) - C(t,x) - \delta\, K(t,x)\right]\right\} dt\,dx \\
& - \frac{1}{L}\int_{-L/2}^{L/2}\int_{t=0}^{t=\infty} Q(t,x)\,K(t,x)\,dx \\
& + \frac{1}{L}\int_0^{\infty} e^{-\rho t}\left[\int_{-L/2}^{L/2}\tau^2\left(Q(t,x)\,\frac{\partial K(t,x)}{\partial x} - \frac{\partial Q(t,x)}{\partial x}\right) K(t,x)\right] dt \\
& + \lim_{t\to\infty}\frac{1}{L}\int_{-L/2}^{L/2} e^{-\rho t}\,\mu(t,x)\,K(t,x)\,dx.
\end{aligned}
$$

If an optimal solution exists, then the variational principle requires that the Gâteaux derivative, evaluated at the optimum, is equal to zero,

$$
\nabla\mathsf{W}[C^*,K^*] = \lim_{\epsilon\to 0}\frac{\mathsf{W}[C(\epsilon),K(\epsilon)] - \mathsf{W}[C^*,K^*]}{\epsilon} = 0.
$$

The Gâteaux derivative is

$$
\begin{aligned}
\nabla\mathsf{W}[C^*,K^*] = {} & \frac{1}{L}\int_{-L/2}^{L/2}\int_0^{\infty} e^{-\rho t}\Big(u'\big(C^*(t,x)\big) - Q(t,x)\Big)\,h_c(t,x) \\
& + e^{-\rho t}\left(\frac{\partial Q(t,x)}{\partial t} + \tau^2\,\frac{\partial^2 Q(t,x)}{\partial x^2} + Q(t,x)\,\Big(f'\big(K^*(t,x)\big) - \rho - \delta\Big)\right) h_k(t,x)\,dt\,dx \\
& + \lim_{t\to\infty}\frac{1}{L}\int_{-L/2}^{L/2} e^{-\rho t}\Big(\mu(t,x) - Q(t,x)\Big)\,h_k(t,x)\,dx \\
& + \frac{1}{L}\int_{-L/2}^{L/2} Q(0,x)\,h_k(0,x)dx \\
& + \frac{1}{L}\int_0^{\infty} e^{-\rho t}\,\tau^2\left[Q\Big(t,\frac{L}{2}\Big)\frac{\partial h_k\big(t,\frac{L}{2}\big)}{\partial x} - \frac{\partial Q\big(t,\frac{L}{2}\big)}{\partial x}\,h_k\Big(t,\frac{L}{2}\Big)\right. \\
& \left. - Q\Big(t,-\frac{L}{2}\Big)\frac{\partial h_k\big(t,-\frac{L}{2}\big)}{\partial x} + \frac{\partial Q\big(t,-\frac{L}{2}\big)}{\partial x}\,h_k\Big(t,-\frac{L}{2}\Big)\right] dt.
\end{aligned}
\tag{5}
$$

We have $\nabla \mathsf{W}[C^*, K^*] = 0$ if and only if all the summands in equation (5) are equal to zero, that is if the following conditions are satisfied: (1) the optimality condition (1); (2) the Euler condiition (2); (3) the dual boundary conditions (3); (4) the initial condition for an admissible perturbation $h_k(0, x) = 0$, for every $x \in \mathrm{X}$; and (5) transversality condition (4). The transversality condition is found by combining $\lim_{t\to\infty}\big(\mu(x,t) - Q(t,x)\big) = 0$ together with the Kuhn-Tucker condition associated to the terminal condition (8). □

Proposition 2. *Assume that $\theta \geq 1$ and $r > \gamma$ and let the initial distribution $k_0(x)$ satisfy conditions (22). Then the coupled system (18)-(19), has a closed form solution, satisfying boundary conditions (5) and (16), for $L = 1$, and the transversality condition (21),*

$$K(t,x) = \int_{-1/2}^{1/2} k_0(y)\, g_k(t, x-y)\, dy \text{ for } (t,x) \in \mathrm{T} \times \mathrm{X}, \tag{6}$$

$$C(t,x) = \int_{-1/2}^{1/2} k_0(y)\, g_c(t, x-y)\, dy \text{ for } (t,x) \in \mathrm{T} \times \mathrm{X}, \tag{7}$$

with kernels

$$g_k(t,x) = \sum_{\omega\in\mathbb{Z}} e^{\lambda_\omega t + i\,2\pi\omega x}$$

$$g_c(t,x) = \sum_{\omega\in\mathbb{Z}} \psi_\omega\, e^{\lambda_\omega t + i\,2\pi\omega x}$$

where $\psi_\omega \equiv r - \gamma + (\theta - 1)\,\xi_\omega > 0$, and $\mathbb{Z} = \{0, \pm1, \pm2, \ldots\}$ is the set of integers and

$$\lambda_\omega \equiv \gamma - \theta\,\xi_\omega, \text{ where } \xi_\omega \equiv \big(2\pi\tau\omega\big)^2 \text{ for } \omega \in \mathbb{Z}. \tag{8}$$

***Proof of Proposition** 2.* We use the finite Fourier transform method to obtain a unique solution to system (18)-(19), over a ring defined over $\mathrm{X} = [-\frac{1}{2}, \frac{1}{2}]$, under the constraints (5), (16), (21) and (22). [31]

[31] We extend the method used in (Butzer and Nessel, 1971, ch. 7) and (Olver, 2014, p. 130). See, in particular, (Butzer and Nessel, 1971, ch. 7) for the conditions under the expressions we found, we obtain actually a solution.

The finite Fourier of $K(t,x)$ and $C(t,x)$ are,

$$\mathcal{K}_\omega(t) = \int_{-1/2}^{1/2} K(t,x)\, e^{-i2\pi\omega x}\, dx, \text{ and } \mathcal{C}_\omega(t) = \int_{-1/2}^{1/2} C(t,x)\, e^{-i2\pi\omega x}\, dx, \text{ for } \omega \in \mathbb{Z}.$$

As

$$\int_{-1/2}^{1/2} \frac{\partial K(t,x)}{\partial t}\, e^{-i2\pi\omega x}\, dx = \mathcal{K}'_\omega(t),$$

and

$$\begin{aligned}\int_{-1/2}^{1/2} \frac{\partial^2 K(t,x)}{\partial x^2}\, e^{-i2\pi\omega x}\, dx &= \int_{-1/2}^{1/2} \frac{\partial K(t,x)}{\partial x}\, e^{-i2\pi\omega x} + (i2\pi\omega) \int_{-1/2}^{1/2} K(t,x)\, e^{-i2\pi\omega x} \\ &+ (i2\pi\omega)^2 \int_{-1/2}^{1/2} K(t,x)\, e^{-i2\pi\omega x}\, dx \\ &= (i2\pi\omega)^2 \mathcal{K}_\omega(t)\end{aligned}$$

from condition (5), and because $e^{i\pi\omega} = e^{-i\pi\omega}$ for $\omega \in \mathbb{Z}$, then the PDE equation (18) can be transformed into an ODE, for each frequency $\omega \in \mathbb{Z}$,

$$\mathcal{K}'_\omega(t) = \big(r - \xi_\omega\big)\, \mathcal{K}_\omega(t) - \mathcal{C}_\omega(t), \text{ for } t \in \mathrm{T} \tag{9}$$

where $\xi_\omega \equiv (2\pi\tau\omega)^2$ is a Fourier mode. Using a similar approach, and noting that

$$\begin{aligned}\int_{-1/2}^{1/2} \frac{1}{C(t,x)} \Big(\frac{\partial C(t,x)}{\partial x}\Big)^2 e^{-i2\pi\omega x}\, dx = \int_{-1/2}^{1/2} \frac{\partial C(t,x)}{\partial x}\, e^{-i2\pi\omega x} + (i2\pi\omega) \int_{-1/2}^{1/2} \frac{\partial C(t,x)}{\partial x}\, e^{-i2\pi\omega x}\, dx = \\ = (i2\pi\omega) \int_{-1/2}^{1/2} C(t,x)\, e^{-i2\pi\omega x} + (i2\pi\omega)^2 \int_{-1/2}^{1/2} C(t,x)\, e^{-i2\pi\omega x}\, dx \\ = (i2\pi\omega)^2\, \mathcal{C}_\omega(t),\end{aligned}$$

from conditions (16). Then the PDE (19) can be transformed into the ODE, for each frequency $\omega \in \mathbb{Z}$,

$$\mathcal{C}'_\omega(t) = \lambda_\omega\, \mathcal{C}_\omega(t), \text{ for } t \in \mathrm{T} \tag{10}$$

where $\lambda_\omega \equiv \gamma - \theta\, \xi_\omega$.

Solving equation (10) yields

$$\mathcal{C}_\omega(t) = \mathcal{C}_\omega(0)\, e^{\lambda_\omega\, t}. \tag{11}$$

Substituting this solution in equation (9) and solving, yields

$$\mathcal{K}_\omega(t) = \left(\mathcal{K}_\omega(0) - \frac{\mathcal{C}_\omega(0)}{\Psi_\omega}\right) e^{(r-\xi_\omega)\,t} + \frac{\mathcal{C}_\omega(0)}{\Psi_\omega}\, e^{\lambda_\omega\, t}, \tag{12}$$

where $\Psi_\omega = r - \xi_\omega - \lambda_\omega = r - \gamma + (\theta - 1)\,\xi_\omega$ which is positive if $r > \gamma$, and $\theta \geq 1$. In the general solution (12), $\mathcal{K}_\omega(0)$ is known because it is the finite spatial Fourier transform of the known initial distribution of capital

$$\mathcal{K}_\omega(0) = \int_{-1/2}^{1/2} k_0(x)\, e^{-i2\pi\omega x}\, dx.$$

However, in both general solutions, (11) and (12), the spatial finite Fourier transform $\mathcal{C}_\omega(0)$ is unknown. As is well known, it can be determined from the transversality condition (21). Because it is formulated in the original location variable, I introduce the conjecture that the transversaiity condition is satisfies if $\mathcal{C}_\omega(0) = \Psi_\omega\, \mathcal{K}_\omega(0)$. Under this conjecture, the particular solution for the spatial finite Fourier transform, for any $t \in \mathrm{T}$, is

$$\begin{aligned} \mathcal{K}_\omega(t) &= \mathcal{K}_\omega(0)\, G_\omega(t) && (13)\\ \mathcal{C}_\omega(t) &= \Psi_\omega\, \mathcal{K}_\omega(0)\, G_\omega(t) && (14) \end{aligned}$$

where $G_\omega(t) = e^{\lambda_\omega\, t}$. Applying the spatial finite Fourier inversion formulas,

$$K(t,x) = \sum_{\omega\in\mathbb{Z}} \mathcal{K}_\omega(t)\, e^{i2\pi\omega x}, \text{ and } C(t,x) = \sum_{\omega\in\mathbb{Z}} \mathcal{C}_\omega(t)\, e^{i2\pi\omega x},$$

we obtain the particular solutions in the original location variable $x \in \mathrm{X}$

$$K(t,x) = \sum_{\omega\in\mathbb{Z}} \mathcal{K}_\omega(0)\, e^{\lambda_\omega\, t}\, e^{i2\pi\omega x} = \sum_{\omega\in\mathbb{Z}} \int_{-1/2}^{1/2} k_0(y)\, e^{-i2\pi\omega y}\, dy\, e^{\lambda_\omega\, t}\, e^{i2\pi\omega x}$$

$$C(t,x) = \sum_{\omega\in\mathbb{Z}} \Psi_\omega \mathcal{K}_\omega(0)\, e^{\lambda_\omega\, t}\, e^{i2\pi\omega x} = \sum_{\omega\in\mathbb{Z}} \Psi_\omega \int_{-1/2}^{1/2} k_0(y)\, e^{-i2\pi\omega y}\, dy\, e^{\lambda_\omega\, t}\, e^{i2\pi\omega x}.$$

An equivalent representation of the solution can be obtained by noting that the inverse (finite) Fourier transform of a product of transformed variables is a convolution of the original variables, then $K(t,x) = \big(K(0) * g_k(t)\big)(x)$ and $C(t,x) = \big(K(0) * g_c(t)\big)(x)$ yielding equations (6) and (7), respectively.

Next I verify that this solution satisfies the boundary and the transversality conditions. First, I show that $K(0,x) = K(0,-x)$ if and only if $\mathcal{K}_\omega(0) = K_{-\omega}(0)$. As $\mathcal{K}_\omega(0) = \int_{-1/2}^{1/2} K(0,x)\, e^{-i2\pi\omega x} dx$ and, defining $\tilde{K}(0,x) = K(0,-x)$ then

$$\int_{-1/2}^{1/2} \tilde{K}(0,x)\, e^{-i2\pi\omega x} dx = \int_{-1/2}^{1/2} K(0,-x)\, e^{-i2\pi(-\omega)(-x)} dx = K_{-\omega}(0).$$

Therefore,

$$K(0,x) = \sum_{\omega\in\mathbb{Z}} \mathcal{K}_\omega(0)\, e^{i2\pi\omega x} = K(0,-x) = \sum_{\omega\in\mathbb{Z}} K_{-\omega}(0)\, e^{i2\pi(-\omega)(-x)}$$

if and only if $\mathcal{K}_\omega(0) e^{i2\pi\omega x} = K_{-\omega}(0) e^{i2\pi(-\omega)(-x)}$. Then, because $\xi_{-\omega} = \xi_\omega$,

$$K(t,-x) = e^{\gamma t} \sum_{\omega\in\mathbb{Z}} K_{-\omega}(0) e^{i2\pi(-\omega)(-x)}\, e^{-\theta\, \xi_{-\omega} t} = e^{\gamma t} \sum_{\omega\in\mathbb{Z}} \mathcal{K}_\omega(0) e^{i2\pi\omega x}\, e^{-\theta\, \xi_\omega t} = K(t,x).$$

Furthermore,

$$\frac{\partial K(t,x)}{\partial x} = e^{\gamma\, t} \sum_{\omega\in\mathbb{Z}} (i2\pi\omega)\, \mathcal{K}_\omega(0)\, e^{i2\pi\omega x - \theta\, \xi_\omega\, t},$$

and as

$$\sum_{\omega\in\mathbb{Z}} (i2\pi\omega)\, \mathcal{K}_\omega(0)\, e^{i2\pi\omega x - \theta\, \xi_\omega\, t} = \sum_{\omega\in\mathbb{Z}} -\, (i2\pi\omega)\, K_{-\omega}(0)\, e^{i2\pi(-\omega)(-x) - \theta\, \xi_{-\omega}\, t},$$

as we are summing over all $\omega \in \mathbb{Z}$, then $\dfrac{\partial K(t,x)}{\partial x} = \dfrac{\partial K(t,-x)}{\partial x}$. Therefore, we have, in particular $K(t,-\frac{1}{2}) = K(t,\frac{1}{2})$ and $\dfrac{\partial K(t,-\frac{1}{2})}{\partial x} = \dfrac{\partial K(t,\frac{1}{2})}{\partial x}$. We can follow a similar approach to show that $C(t,-\frac{1}{2}) = C(t,\frac{1}{2})$ and $\dfrac{\partial C(t,-\frac{1}{2})}{\partial x} = \dfrac{\partial C(t,\frac{1}{2})}{\partial x}$ if we observe that $\Psi_\omega = \Psi_{-\omega}$.

From Corollary 2

$$\lim_{t\to\infty} C(t,x)^{-\theta}\, K(t,x)\, e^{-\rho t} = \lim_{t\to\infty} (r-\gamma)\mathsf{M}[K](0) e^{-\left(\rho+(\theta-1)\,\gamma\right)t} = 0$$

as $\rho + (\theta-1)\,\gamma > 0$. Then transversality condition (21) is satisfied by solutions (6) and (7), and, therefore, by the transformed particular solutions (13) and (14). □

Corollary 1. *The solution satisfies a conservation property, for every points in time, in the following sense,*

$$\mathrm{M}[K(t)] = \int_{-1/2}^{1/2} K(t,x)\, dx = e^{\gamma t}\, \mathrm{M}[K(0)],\ \textit{for any}\, t \in \mathrm{T}$$

and

$$\mathrm{M}[C(t)] = \int_{-1/2}^{1/2} C(t,x)\, dx = e^{\gamma t}\, (r-\gamma)\, \mathsf{M}[K](0),\ \textit{for any}\, t \in \mathrm{T}$$

Proof of Corollary 1. The aggregate capital stock for any time $t \in \mathrm{T}$ is

$$\begin{aligned}
\mathsf{M}[K(t)] &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} k_0(y)\, g(t, x-y)\, dy\, dx \\
&= \int_{-1/2}^{1/2} k_0(y) \left(\int_{-1/2}^{1/2} g(t, x-y)\, dx \right) dy \\
&= \int_{-1/2}^{1/2} k_0(y) \left(\int_{-1/2-y}^{1/2-y} g(t, z)\, dz \right) dy \\
&= \int_{-1/2}^{1/2} k_0(y) \left(\int_{-1/2}^{1/2} g(t, z)\, dz \right) dy \\
&= \int_{-1/2}^{1/2} k_0(y)\, dy \int_{-1/2}^{1/2} g(t, z)\, dz \\
&= \mathsf{M}[K](0) \int_{-1/2}^{1/2} g(t, x)\, dx
\end{aligned}$$

the third passage is valid because $g(t, x)$ is a periodic function with period equal to one. Observing that $\mathbb{Z} = \mathbb{Z}_- \cup \{0\} \cup \mathbb{Z}_+$, we obtain

$$\begin{aligned}
\int_{-1/2}^{1/2} g(t, x)\, dx &= \int_{-1/2}^{1/2} \sum_{\omega \in \mathbb{Z}} e^{(\gamma - \theta \xi_\omega)\, t}\, e^{i 2\pi \omega x}\, dx \\
&= e^{\gamma t} + \int_{-1/2}^{1/2} \Big(\sum_{\omega \in \mathbb{Z}_-} e^{(\gamma - \theta \xi_\omega)\, t}\, e^{i 2\pi \omega x} + \sum_{\omega \in \mathbb{Z}_+} e^{(\gamma - \theta \xi_\omega)\, t}\, e^{i 2\pi \omega x} \Big)\, dx \\
&= e^{\gamma t} + \int_{-1/2}^{0} + \int_{0}^{1/2} \Big(\sum_{\omega \in \mathbb{Z}_-} e^{(\gamma - \theta \xi_\omega)\, t}\, e^{i 2\pi \omega x} + \sum_{\omega \in \mathbb{Z}_+} e^{(\gamma - \theta \xi_\omega)\, t}\, e^{i 2\pi \omega x} \Big)\, dx \\
&= e^{\gamma t}
\end{aligned}$$

because

$$\begin{aligned}
&\int_{-1/2}^{0} \sum_{\omega \in \mathbb{Z}_-} e^{(\gamma - \theta \xi_\omega)\, t}\, e^{i 2\pi \omega x}\, dx + \int_{0}^{1/2} \sum_{\omega \in \mathbb{Z}_+} e^{(\gamma - \theta \xi_\omega)\, t}\, e^{i 2\pi \omega x}\, dx \\
&\qquad = -\Big(\int_{-1/2}^{0} \sum_{\omega \in \mathbb{Z}_+} e^{(\gamma - \theta \xi_\omega)\, t}\, e^{i 2\pi \omega x}\, dx + \int_{0}^{1/2} \sum_{\omega \in \mathbb{Z}_-} e^{(\gamma - \theta \xi_\omega)\, t}\, e^{i 2\pi \omega x}\, dx \Big).
\end{aligned}$$

$\square$

Corollary 2. *Solutions (6) and (7) converge asymptotically to an exponentially growing flat distributions*

$$\lim_{t\to\infty} K(t,x) = \mathsf{M}[K](0)\, e^{\gamma t},$$
$$\lim_{t\to\infty} C(t,x) = (r-\gamma)\,\mathsf{M}[K](0)\, e^{\gamma t}.$$

Proof of Corollary 2. Equation (6) is equivalent to

$$\begin{aligned} e^{-\gamma t}\, K(t,x) &= \sum_{\omega\in\mathbb{Z}} \left(\int_{-1/2}^{1/2} k_0(y) e^{i2\pi\omega(x-y)} dy \right) e^{-\theta\xi_\omega t} \\ &= \int_{-1/2}^{1/2} k_0(y)dy + \sum_{\omega\in\mathbb{Z}/\{0\}} \left(\int_{-1/2}^{1/2} k_0(y) e^{i2\pi\omega(x-y)} dy \right) e^{-\theta\xi_\omega t}. \end{aligned}$$

As $\lim_{t\to\infty} e^{-\theta\xi_\omega t} = 0$, for any $\omega \in \mathbb{Z}/\{0\}$, then $\lim_{t\to\infty} e^{-\gamma t}\, K(t,x) = \int_{-1/2}^{1/2} k_0(y)dy = \mathsf{M}[K](0)$. Using an analogous approach, equation (7) is equivalent to

$$e^{-\gamma t}\, C(t,x) = (r-\gamma) \int_{-1/2}^{1/2} k_0(y)dy + \sum_{\omega\in\mathbb{Z}/\{0\}} \left(\int_{-1/2}^{1/2} k_0(y) e^{i2\pi\omega(x-y)} dy \right) (r-\gamma+(\theta-1)\xi_\omega)\, e^{-\theta\xi_\omega t}.$$

Again, as $\lim_{t\to\infty} e^{-\theta\xi_\omega t} = 0$, then $\lim_{t\to\infty} e^{-\gamma t}\, C(t,x) = (r-\gamma)\,\mathsf{M}[K](0)$. $\square$

The above results apply to the case in which $L = \infty$ and $X = \mathbb{R}$ if the initial distribution satisfies

$$\begin{cases} \lim_{x\to-\infty} k_0(x) = \lim_{x\to\infty} k_0(x), \text{ and } \lim_{x\to-\infty} \frac{\partial k_0(x)}{\partial x} = \lim_{x\to\infty} \frac{\partial k_0(x)}{\partial x} \\ \mathsf{M}[K](0) = \int_{-\infty}^{\infty} k_0(x)\, dx < \infty \end{cases} \tag{15}$$

This allows us to obtain a more specific (formal) closed form solution involving Gaussian kernels:[32]

[32]This is well known from Brito (2011), and Boucekkine et al. (2013).

Proposition 3. *Assume an extension to the previous social welfare problem where $L \to \infty$ with the initial distribution $k_0(x)$ satisfying conditions (15). Then the problem has a (formal) closed form solution that takes the form of an exponential trend multiplied by a detrended time varying density*

$$K(t,x) = e^{\gamma t}\, k(t,x), \text{ and } C(t,x) = e^{\gamma t}\, c(t,x), \text{ for each } (t,x) \in \mathrm{T} \times \mathrm{X}$$

where the detrended distribution for capital is

$$k(t,x) = \int_{-\infty}^{\infty} k_0(\xi)\, \frac{e^{-\frac{(x-\xi)^2}{2\,\sigma(t)^2}}}{\sqrt{2\pi}\,\sigma(t)}\, d\xi, \text{ for } (t,x) \in \mathrm{T} \times \mathrm{X}, \tag{16}$$

with the time-varying standard deviation $\sigma(t) \equiv \tau\sqrt{2\,\theta\,t} \geq 0$, and the detrended distribution for consumption is

$$c(t,x) = \int_{-\infty}^{\infty} k_0(\xi)\, \psi(t, x-\xi)\, \frac{e^{-\frac{(x-\xi)^2}{2\,\sigma(t)^2}}}{\sqrt{2\pi}\,\sigma(t)}\, d\xi, \text{ for } (t,x) \in \mathrm{T} \times \mathrm{X} \tag{17}$$

where

$$\psi(t,x) \equiv r - \gamma + (1-\theta)\,\tau^2 \left(\frac{x^2 - \sigma(t)^2}{\sigma(t)^4} \right). \tag{18}$$

Proof of Proposition 3. The distributional MHDS (18)-(19) is a system of non-linear parabolic partial differential equations (PDE). In order to obtain a general solution, I introduce the trial functions (ansatz)[33],

$$C(t,x) = \mathcal{C}(t,\omega)e^{2\pi i\omega x}, \; K(t,x) = \mathcal{K}(t,\omega)e^{2\pi i \omega x}, \text{for each } (t,x) \in \mathrm{T} \times \mathrm{X}$$

where $i = \sqrt{-1}$ and $\omega \in \mathbb{R}$ is a wave number for the propagation of impulses across space. This transformation is valid for every point $x \in \mathrm{X}$ because X is an unbounded open set.

[33] This is an approach followed in applied literature, example Cross and Greenside (2009).

System (18)-(19) is transformed from a PDE system for $(K(t,x), C(t,x))$ over (t,x) into a ODE system for $(\mathcal{C}(t,\omega), \mathcal{K}(t,\omega))$ over t parameterized by ω

$$\frac{\partial \mathcal{K}(t,\omega)}{\partial t} = \left(r - (2\tau\pi\omega)^2\right) \mathcal{K}(t,\omega) - \mathcal{C}(t,\omega), \tag{19}$$
$$\frac{\partial \mathcal{C}(t,\omega)}{\partial t} = \left(\gamma - \theta\,(2\tau\pi\omega)^2\right) \mathcal{C}(t,\omega). \tag{20}$$

If the initial condition on $\mathcal{K}(0,\omega)$ and $\mathcal{C}(0,\omega)$ allow for the satisfaction of the initial and transversality condition on the original coordinates, implies that the ODE system is well posed. We conjecture that there is a linear time-independent but frequency-dependent relationship between consumption and the capital stock: $\mathcal{C}(t,\omega) = \mathcal{M}(\omega)\,\mathcal{K}(t,\omega)$. If his conjecture if right it provides a necessary condition for the satisfaction of the transversality condition. It also implies that $\partial\mathcal{C}(t,\omega)/\partial t = \mathcal{M}(\omega)\partial\mathcal{K}(t,\omega)/\partial t$. Substituting in equations (20) and (19) we find that our conjecture is right if and only if $\mathcal{M}(\omega) = r - \gamma + (\theta - 1)(2\tau\pi\omega)^2$.

Substituting $\mathcal{C}(t,\omega)$ in equation (19) we obtain a linear ODE for $\mathcal{K}(t,\omega)$,

$$\frac{\partial \mathcal{K}(t,\omega)}{\partial t} = \lambda(\omega)\,\mathcal{K}(t,\omega),$$

where the coefficient $\lambda(\omega) \equiv \gamma - \theta(2\tau\pi\omega)^2$ is an eigenfunction, or characteristic exponent. This equation has the solution

$$\mathcal{K}(t,\omega) = \mathcal{K}(0,\omega)\,G_k(t,\omega),$$

with the Green's function $G_k(t,\omega) \equiv e^{\lambda(\omega)t}$. Therefore, the solution for consumption is

$$\mathcal{C}(t,\omega) = \mathcal{K}(0,\omega)\,\mathcal{M}(\omega)\,e^{\lambda(w)t}.$$

If $\mathcal{K}(t,\omega)$ and $\mathcal{C}(t,\omega)$ are taken as the spatial Fourier transforms on $L^2(\mathbb{R})$ in its symmetric form (see (Goldberg, 1962, p.43) or Kammler (2000)),

$$\mathcal{K}(t,\omega) = \mathcal{F}\big[K(t,x)\big] = \int_{-\infty}^{\infty} K(t,x)\, e^{-2\pi i\omega x}\, dx$$

and

$$\mathcal{C}(t,\omega) = \mathcal{F}\big[C(t,x)\big] = \int_{-\infty}^{\infty} C(t,x)\, e^{-2\pi i\omega x}\, dx,$$

we can recover $K(t,x)$ and $C(t,x)$ as the inverse spatial Fourier transforms

$$K(t,x) = \mathcal{F}^{-1}\big[\mathcal{K}(t,\omega)\big] = \int_{-\infty}^{\infty} \mathcal{K}(t,\omega)\, e^{2\pi i\omega x}\, d\omega = \int_{-\infty}^{\infty} \mathcal{K}(0,\omega)\, G_k(t,\omega)\, e^{2\pi i\omega x}\, d\omega$$

and

$$C(t,x) = \mathcal{F}^{-1}\big[\mathcal{C}(t,\omega)\big] = \int_{-\infty}^{\infty} \mathcal{C}(t,\omega)\, e^{2\pi i\omega x}\, d\omega = \int_{-\infty}^{\infty} \mathcal{K}(0,\omega)\, G_c(t,\omega)\, e^{2\pi i\omega x}\, d\omega.$$

For any $t > 0$ we have

$$K(t,x) = \mathcal{F}^{-1}[\mathcal{K}(0,\omega)\, G_k(t,\omega)] = k_0(x) * g_k(t,x) = \int_{-\infty}^{\infty} k_0(\xi)\, g_k(t,x-\xi)\, d\xi, \text{ for } t > 0$$

where

$$k_0(x) = K(0,x) = \mathcal{F}^{-1}\big[\mathcal{K}(0,\omega)\big] = \int_{-\infty}^{\infty} \mathcal{K}(0,\omega)\, e^{2\pi i\omega x} d\omega$$

and

$$g_k(t,x) = \mathcal{F}^{-1}[G_k(t,\omega)] = \int_{-\infty}^{\infty} e^{\lambda(\omega)\, t + 2\pi i x\omega}\, d\omega = e^{\gamma t}\, g_k^d(t,x)$$

is a Gaussian kernel, where

$$g_k^d(t,x) \equiv \frac{e^{-\frac{x^2}{2\,\sigma(t)^2}}}{\sqrt{2\pi}\,\sigma(t)}.$$

The detrended kernel has a normal distribution $g_k^d(t,x) \sim N(0,\sigma(t)$. The standard deviation $\sigma(t) \equiv \tau\sqrt{2\,\theta\, t}$, is a positive, increasing, and concave function of time, which yields equation (16).

The solution for consumption, taking $C(t,x) = \mathcal{F}^{-1}[\mathcal{K}(0,\omega)\, G_c(t,\omega)]$ is

$$C(t,x) = k_0(x) * g_c(t,x) = \int_{-\infty}^{\infty} k_0(\xi)\, g_c(t, x-\xi) d\xi, \text{ for } t \geq 0$$

where $g_c(t,x) = \mathcal{F}^{-1}[\mathcal{M}(\omega)\, G_k(t,\omega)] = m(x) * g_k(t,x)$, where

$$m(x) = \mathcal{F}^{-1}[\mathcal{M}(\omega)] = (r-\gamma)\delta(x) + (1-\theta)\, \frac{(2\pi\tau)^2}{(2\pi)^2}\, \delta''(x)\big)$$

where $\delta(\cdot)$ and $\delta''(\cdot)$ is the Dirac's delta "function" and its second derivative. Therefore, the consumption kernel is

$$\begin{aligned} g_c(t,x) &= \int_{-\infty}^{\infty} \Big((r-\gamma)\delta(\xi) + (1-\theta)\, \frac{(2\pi\tau)^2}{(2\pi)^2}\, \delta''(\xi)\Big)\, g_k(t, x-\xi)\, d\xi \\ &= \int_{-\infty}^{\infty} \Big((r-\gamma) g_k(t, x-\xi) + (1-\theta)\, \tau^2\, \frac{\partial^2 g_k(t, x-\xi)}{\partial x^2}\Big)\, \delta(\xi)\, d\xi \\ &= (r-\gamma) g_k(t,x) + (1-\theta)\, \tau^2\, \frac{\partial^2 g_k(t,x)}{\partial x^2}. \end{aligned}$$

Substituting the kernel for capital, yields equation (17),

$$\begin{aligned} g_c(t,x) &= e^{\gamma t} \left[r - \gamma + (1-\theta)\, \tau^2 \Big(\frac{x^2 - \sigma(t)^2}{\sigma(t)^4} \Big) \right] \frac{e^{-\frac{x^2}{2\, \sigma(t)^2}}}{\sqrt{2\pi}\, \sigma(t)} \\ &= e^{\gamma t}\, \psi(t,x)\, g^d_k(t,x) \\ &= e^{\gamma t}\, g^d_c(t,x), \end{aligned}$$

where function $g^d_c(t,x) = \psi(t,x)\, g^d_k(t,x)$ is the detrended consumption kernel. We can write $g^d_c(t,x) = \big((r-\gamma)\, H_0(x) + (1-\theta)\, (2\pi\tau)^2\, H_2(x)\big)\, g^d_k(t,x)$ where $H_0(\cdot)$ and $H_2(\cdot)$ are Hermite monomials of order 0 and 2 associated to the Gaussian kernel $g^d_k(\cdot, x)$. Because the Gaussian kernel "dominates" the Hermite monomials when $|x| \to \infty$ then $\lim_{t\to 0} g_c(t,x) = (r-\gamma)\, \delta(x)$, where $\delta(.)$ is Dirac's delta generalized function.

From our assumption that $\mathsf{M}[K](0) < \infty$, and form the properties of the Gaussian distribution we find that the detrended distribution converges to the bounded spatial-independent aggregate $\mathsf{M}[K](0)$

$$\lim_{t\to\infty} k(t,x) = \lim_{t\to\infty} \int_{-\infty}^{\infty} k_0(y)\, g_k^d(t,x-y)\, dy = \mathsf{M}[K](0)$$

and consumption converges also to a bounded, space-independent constant

$$\lim_{t\to\infty} c(t,x) = \lim_{t\to\infty} \int_{-\infty}^{\infty} k_0(y)\, g_c^d(t,x-y)\, dy = (r-\gamma)\, \mathsf{M}[K](0),$$

which is positive, under the assumption $r > \gamma$. This implies that the transversality condition (21) is also satisfied, similarly to the bounded case. □

Proposition 4. *The particular of PDE the system,* (36)-(37)*, satisfying conditions* (38)*,* (39)*, and* (40) *is*

$$\begin{aligned} \tilde{K}(t,x) &= \int_{-1/2}^{1/2} \tilde{K}(0,y)\, g_k\big(t,x-y\big)\, dy, \ \textit{for } (t,x) \in \mathrm{T}\times\mathrm{X}, \\ \tilde{Q}(t,x) &= \int_{-1/2}^{1/2} \tilde{K}(0,y)\, g_q\big(t,x-y\big)\, dy, \ \textit{for } (t,x) \in \mathrm{T}\times\mathrm{X}, \end{aligned}$$

with kernels

$$\begin{aligned} g_k(t,x) &= \sum_{\omega\in\mathbb{Z}} e^{\lambda_\omega^-\, t + i2\pi\omega x}, \\ g_q(t,x) &= u''(Q^*) \sum_{\omega\in\mathbb{Z}} \big(\lambda_\omega^+ - \xi_\omega\big)\, e^{\lambda_\omega^-\, t+i2\pi\omega x}, \end{aligned}$$

where ξ_ω is defined in equation (8)*, and*

$$\begin{aligned} \lambda_\omega^- &= \frac{\rho}{2} - \sqrt{\Delta_\omega} \equiv \frac{\rho}{2} - \left(\left(\frac{\rho}{2} - \xi_\omega\right)^2 + \mu\right)^{\frac{1}{2}},\ \omega\in\mathbb{Z}, \\ \lambda_\omega^+ &= \frac{\rho}{2} + \sqrt{\Delta_\omega} \equiv \frac{\rho}{2} + \left(\left(\frac{\rho}{2} - \xi_\omega\right)^2 + \mu\right)^{\frac{1}{2}},\ \omega\in\mathbb{Z}, \end{aligned}$$

where $\lambda_\omega^- + \lambda_\omega^+ = \rho > 0$.

Proof of Proposition 4. Again I apply discrete (or finite) spatial Fourier transforms to

$\tilde{K}(t,x)$ and $\tilde{Q}(t,x)$ taking t parametrically. Thus, the variational PDE system (36)-(37) is transformed into a linear ODE system, for every frequency $\omega \in \mathbb{Z}$, in matrix notation

$$\begin{pmatrix} \mathcal{K}'_\omega(t) \\ \mathcal{Q}'_\omega(t) \end{pmatrix} = \mathbf{J}_\omega \begin{pmatrix} \mathcal{K}_\omega(t) \\ \mathcal{Q}_\omega(t) \end{pmatrix}, \text{ where } \mathbf{J}_\omega \equiv \begin{pmatrix} \rho - \xi_\omega & -\big(u''(Q^*)\big)^{-1} \\ -Q^* f''(K^* & \xi_\omega \end{pmatrix}, \text{ for each } \omega \in \mathbb{Z}, \tag{21}$$

and $\xi_\omega \equiv \big(2\pi\tau\omega\big)$. The trace and the determinant of matrix $(\mathbf{J}_\omega)$ are $\mathrm{tr}(\mathbf{J}_\omega) = \rho > 0$, and $\det(\mathbf{J}_\omega) = \big(\rho - \xi_\omega\big)\,\xi_\omega - \mu$, respectively, the discriminant is $\Delta(\mathbf{J}_\omega) = \left(\frac{\rho}{2} - \xi_\omega\right)^2 + \mu > 0$. Then the eigenfunctions of matrix $\mathbf{J}_\omega$, are $\lambda^\pm_\omega = \dfrac{\rho}{2} \pm \sqrt{\Delta_\omega}$.

We readily see that: first, they are both real for all frequencies $\omega \in \mathbb{Z}$, (ii) eigenfunction λ^+_ω is stricly positive for all frequencies $\omega \in \mathbb{Z}$; and (iii) the eigenfunction λ^-_ω has an ambiguous sign, depending on value of the parameters ρ and μ, and on the frequency $\omega \in \mathbb{Z}$. However, the "stable" eigenfunction λ^-_ω is negative for $\omega = 0$ and for $|\omega| = \infty$: that is $\lambda^-_0 < 0$ and $\lim_{\omega \to \pm\infty} \lambda^-_\omega < 0$.

As both eigenfunctions are real numbers, for any $\omega \in \mathbb{Z}$, then the Jordan canonical form of Jacobian $\mathbf{J}_\omega$ is $\Lambda(\omega) = \mathbf{V}^{-1}_\omega\,\mathbf{J}_\omega\,\mathbf{V}_\omega$ where $\lambda_\omega = \mathrm{diag}\big(\lambda_+(\omega), \lambda_-(\omega)\big)$, and $\mathbf{V}_\omega = \big(\mathbf{V}^+_\omega | \mathbf{V}^-_\omega\big)$ is the eigenvector matrix associated to the frequency $\omega \in \mathbb{Z}$. An elementary computation yields $\mathbf{V}^i_\omega = (1, \psi^i_\omega)^\top$ where

$$\psi^i_\omega \equiv u''(C^*)\left(\rho - \xi_\omega - \lambda^i_\omega\right) = u''(C^*)\left(\lambda^{-i}_\omega - \xi_\omega\right), \text{ for } i = \pm.$$

Then, the general solution of system (21),

$$\begin{pmatrix} \mathcal{K}_\omega(t) \\ \mathcal{Q}_\omega(t) \end{pmatrix} = z^+_\omega \begin{pmatrix} 1 \\ \psi^+_\omega \end{pmatrix} e^{\lambda^+_\omega t} + z^-_\omega \begin{pmatrix} 1 \\ \psi^-_\omega \end{pmatrix} e^{\lambda^-_\omega t}, \text{ for } (t,\omega) \in \mathrm{T} \times \mathbb{Z}$$

is a weighted sum of the two growth factors $e^{\lambda^+_\omega t}$ and $e^{\lambda^-_\omega t}$, where $z^\pm_\omega$ are two arbitrary scalar functions of ω. As $\lambda_+(\omega) > \rho > 0$, for all $\omega \in \mathbb{Z}$, then a necessary condition for the verification of the transversality condition is that $z^+_\omega = 0$ identically for all $\omega \in \mathbb{Z}$. The particular solution satisfying the initial condition is determine by setting $z^-_\omega = \mathcal{K}_\omega(0)$, which is the discrete spatial Fourier transform of the initial perturbation $\tilde{K}(0,x)$, $\mathcal{K}_\omega(0) = \int_{-1/2}^{1/2} \tilde{K}(0,x)\, e^{-i2\pi\omega x}\, dx$. Then, the particular solution of the system is

$$\begin{aligned} \mathcal{K}_\omega(t) &= \mathcal{K}_\omega(0)\, \mathcal{G}_{k,\omega}(t) \\ \mathcal{Q}_\omega(t) &= \mathcal{K}_\omega(0)\, \psi^-_\omega\, \mathcal{G}_{k,\omega}(t), \end{aligned} \tag{22}$$

where $\mathcal{G}_{k,\omega} \equiv e^{-\lambda^-_\omega t}$ and, as $u''(C^*) < 0$ then

$$\psi^-_\omega = u''(C^*)\left(\lambda^+_\omega - \xi_\omega\right) = u''(C^*)\left(\frac{\rho}{2} - \xi_\omega + \left(\left(\frac{\rho}{2} - \xi_\omega\right)^2 + \mu\right)^{\frac{1}{2}}\right) < 0, \text{ for each } \omega \in \mathbb{Z}. \tag{23}$$

Applying the inverse spatial Fourier transform we finally obtain

$$\tilde{K}(t,x) = \sum_{\omega\in\mathbb{Z}} \mathcal{K}_\omega(0)\, \mathcal{G}_{k,\omega}(t)\, e^{i2\pi\omega x} \tag{24}$$

$$\tilde{Q}(t,x) = \sum_{\omega\in\mathbb{Z}} \mathcal{K}_\omega(0)\, \psi^-_\omega\, \mathcal{G}_{k,\omega}(t)\, e^{i2\pi\omega x}. \tag{25}$$

As

$$\begin{aligned} Q^*\tilde{K}(t,x) + K^*\tilde{Q}(t,x) &= \sum_{\omega\in\mathbb{Z}} \left(Q^*\, \mathcal{K}_\omega(t) + K^*\, \mathcal{Q}_\omega(t)\right) e^{i2\pi\omega x} \\ &= \sum_{\omega\in\mathbb{Z}} \left(Q^* + \psi^-_\omega\, K^*\right) \mathcal{K}_\omega(0)\, e^{i2\pi\omega x}\, e^{\lambda^-_\omega t} \end{aligned}$$

then

$$\lim_{t\to\infty} e^{-\rho t}\left(Q^*\tilde{K}(t,x) + K^*\tilde{Q}(t,x)\right) = \lim_{t\to\infty} \sum_{\omega\in\mathbb{Z}} \left(Q^* + \psi^-_\omega\, K^*\right) \mathcal{K}_\omega(0)\, e^{i2\pi\omega x}\, e^{-\lambda^+_\omega t} = 0$$

because $\lambda^+_\omega > 0$ for every $\omega \in \mathbb{Z}$, and the set X is bounded. Therefore, the transversality condition holds. Substituting $\mathcal{K}_\omega(0)$, we can represent the solutions (24) and (25) using singular integrals as in equations (42) and (43). □

Corollary 3. *The solution satisfies a conservation property, for every points in time, in the following sense,*

$$\mathrm{M}[K(t)] = \int_{-1/2}^{1/2} K(t,x)\, dx = e^{\gamma t}\, \mathrm{M}[K(0)], \; \textit{for any}\, t \in \mathrm{T}$$

and

$$\mathrm{M}[C(t)] = \int_{-1/2}^{1/2} C(t,x)\, dx = e^{\gamma t}\, (r-\gamma)\, \mathsf{M}[K](0), \; \textit{for any}\, t \in \mathrm{T}$$

Proof of Corollary 3. Expanding equation (24) we have

$$\begin{aligned}
\tilde{K}(t,x) &= \sum_{\omega \in \mathbb{Z}} \Big(\int_{-1/2}^{1/2} \tilde{K}(0,y)\, e^{i2\pi\omega(x-y)}\, dy \Big)\, e^{\lambda_\omega^- t} \\
&= e^{\lambda_0^- t} \int_{-1/2}^{1/2} \tilde{K}(0,y)\, dy + \sum_{\omega \in \mathbb{Z}/\{0\}} \Big(\int_{-1/2}^{1/2} \tilde{K}(0,y)\, e^{i2\pi\omega(x-y)}\, dy \Big)\, e^{\lambda_\omega^- t} \\
&= e^{\lambda_0^- t}\, \mathsf{M}[\tilde{K}](0) + \sum_{\omega \in \mathbb{Z}/\{0\}} \Big(\int_{-1/2}^{1/2} \tilde{K}(0,y)\, e^{i2\pi\omega(x-y)}\, dy \Big)\, e^{\lambda_\omega^- t}
\end{aligned}$$

Therefore:

1. if there is spectral stability, that is $\lambda_\omega^- < 0$ for all $\omega \in \mathbb{Z}$, then the second term in the last expression will converge to zero when $t \to \infty$ and

$$\lim_{t\to\infty} \tilde{K}(t,x) = \lim_{t\to\infty} \mathsf{M}[\tilde{K}](0) e^{\lambda_0^- t} = 0, \; \text{for all}\, x \in \mathrm{X}$$

2. if there is spectral instability, it is convenient to write

$$\begin{aligned}
\tilde{K}(t,x) = e^{\lambda_0^- t}\, \mathsf{M}[\tilde{K}](0) &+ \sum_{\omega \in \mathbb{Z}^s} \Big(\int_{-1/2}^{1/2} \tilde{K}(0,y)\, e^{i2\pi\omega(x-y)}\, dy \Big)\, e^{\lambda_\omega^- t} \\
&+ \sum_{\omega \in \mathbb{Z}^u} \Big(\int_{-1/2}^{1/2} \tilde{K}(0,y)\, e^{i2\pi\omega(x-y)}\, dy \Big)\, e^{\lambda_\omega^- t}.
\end{aligned}$$

Taking the limit $t \to \infty$, the first two terms converge to zero but the last term becomes unbounded almost everywhere. If the $\tilde{K}(0,x)$ is an even function then there are two values $\tilde{x}$ and $-\tilde{x}$ in X which introduce a partition in X, say between two open sets X^+ and X^-, such that

$$\lim_{t\to\infty} \tilde{K}(t,x) = \begin{cases} +\infty & \text{if } x \in X^+ \\ 0 & \text{if } x \in \{-\tilde{x}, \tilde{x}\} \\ -\infty & \text{if } x \in X^- \end{cases}$$

3. however, if $\tilde{K}(0,x)$ is a bounded even function then

$$\lim_{t\to\infty} \mathsf{M}[\tilde{K}](t) = \lim_{t\to\infty} \int_{-1/2}^{1/2} \tilde{K}(t,x)\,dx = \\ e^{\lambda_0^- t}\,\mathsf{M}[\tilde{K}](0) \\ + \sum_{\omega\in\mathbb{Z}/\{0\}} \Big(\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \tilde{K}(0,y)\,e^{i2\pi\omega(x--y)}\,dy\,dx \Big)\, e^{\lambda_\omega^- t} = 0$$

for any λ_ω^-.

Using the same approach for equation (25) we can expand $\tilde{Q}(t,x)$ to

$$\begin{aligned} \tilde{Q}(t,x) &= \sum_{\omega\in\mathbb{Z}} \psi_\omega^- \Big(\int_{-1/2}^{1/2} \tilde{K}(0,y)\,e^{i2\pi\omega(x-y)}\,dy \Big)\, e^{\lambda_\omega^- t} \\ &= e^{\lambda_0^- t}\,u''(C^*)\,\lambda_0^+ \int_{-1/2}^{1/2} \tilde{K}(0,y)\,dy + \sum_{\omega\in\mathbb{Z}/\{0\}} \psi_\omega^- \Big(\int_{-1/2}^{1/2} \tilde{K}(0,y)\,e^{i2\pi\omega(x-y)}\,dy \Big)\, e^{\lambda_\omega^- t} \\ &= e^{\lambda_0^- t}\,u''(C^*)\,\lambda_0^+\,\mathsf{M}[\tilde{K}](0) + \sum_{\omega\in\mathbb{Z}/\{0\}} \psi_\omega^- \Big(\int_{-1/2}^{1/2} \tilde{K}(0,y)\,e^{i2\pi\omega(x-y)}\,dy \Big)\, e^{\lambda_\omega^- t}. \end{aligned}$$

Therefore, we have:

1. if there is spectral stability,

$$\lim_{t\to\infty} \tilde{Q}(t,x) = \lim_{t\to\infty} u''(C^*)\,\lambda_0^+\,\mathsf{M}[\tilde{K}](0) e^{\lambda_0^- t} = 0, \text{ for all } x \in X$$

2. if there is spectral instability,

$$\lim_{t\to\infty} \tilde{Q}(t,x) = \begin{cases} -\infty & \text{if } x \in \mathrm{X}^+ \\ 0 & \text{if } x \in \{ -\tilde{x}, \tilde{x} \} \\ +\infty & \text{if } x \in \mathrm{X}^- \end{cases}$$

3. for any case, if the initial perturbation $\tilde{K}(0,x)$ is an even function

$$\lim_{t\to\infty} \mathsf{M}[\tilde{Q}](t) = \lim_{t\to\infty} u''(C^*)\,\lambda_0^+\,\mathsf{M}[\tilde{K}](0)e^{\lambda_0^-\,t} = 0.$$

□

Proposition 5. *Consider an economy in which the local dynamics from a flat steady state is represented by PDE system (36)-(37), and assume that it is distributionaly stable. Then, the short-run multipliers for a non-anticipated, permanent and spatially-heterogeneous productivity shock $\tilde{A}(x)$ are*

$$\begin{aligned} \tilde{K}(t,x) &= \bar{K}(x) - \int_{-1/2}^{1/2} \tilde{A}(y)\,m^k(t,x-y)\,dy,\ \textit{for each } (t,x) \in \mathrm{T}\times\mathrm{X} \\ \tilde{Q}(t,x) &= \bar{Q}(x) - \int_{-1/2}^{1/2} \tilde{A}(y)\,m^q(t,x-y)\,dy,\ \textit{for each } (t,x) \in \mathrm{T}\times\mathrm{X} \end{aligned}$$

where the long-run multipliers are

$$\begin{aligned} \bar{K}(x) &= \int_{-1/2}^{1/2} \tilde{A}(y)\,\bar{m}^k(x-y)\,dy,\ \textit{for each } x \in \mathrm{X} \\ \bar{Q}(x) &= \int_{-1/2}^{1/2} \tilde{A}(y)\,\bar{m}^q(x-y)\,dy,\ \textit{for each } x \in \mathrm{X} \end{aligned}$$

where

$$\bar{m}^k(x) = \sum_{\omega\in\mathbb{Z}} \mathcal{M}^k_\omega\,e^{i2\pi\omega x}\ \textit{and}\ \bar{m}^q(x) = \sum_{\omega\in\mathbb{Z}} \mathcal{M}^q_\omega\,e^{i2\pi\omega x},$$

and

$$m^k(t,x) = \sum_{\omega\in\mathbb{Z}} \mathcal{M}^k_\omega\, e^{i2\pi\omega x + \lambda^-_\omega\, t}, \textit{ and } m^q(t,x) = \sum_{\omega\in\mathbb{Z}} \psi^-_\omega \mathcal{M}^k_\omega\, e^{i2\pi\omega x + \lambda^-_\omega\, t},$$

where ψ^-_ω *and* λ^-_ω *are given in equations* (23) *and* (46)*, respectively, and*

$$\mathcal{M}^k_\omega \equiv -\frac{\xi_\omega\, f_A(K^*) - \frac{Q^*}{u''(C^*)}\, f_{AK}(K^*)}{\xi_\omega\,(\rho - \xi_\omega) - \mu}, \textit{ and } \mathcal{M}^q_\omega \equiv -\frac{Q^*\left(f_A(K^*)\, f_{KK}(K*) + (\xi_\omega - \rho) f_{AK}(K^*)\right)}{\xi_\omega\,(\rho - \xi_\omega) - \mu}.$$

Proof of Proposition 5*.* If we introduce an anticipated, permanent and spatially-heterogeneous change in productivity, $\tilde{A}(x)$, from A_0 to $A_1(x)$. the linearized PDE system changes from (36)-(37) to

$$\frac{\partial \tilde{K}(t,x)}{\partial t} = \tau^2\, \frac{\partial^2 \tilde{K}(t,x)}{\partial x^2} + \rho\, \tilde{K}(t,x) - \left(u''(Q^*)\right)^{-1} \tilde{Q}(t,x) + f_A(K^*)\, \tilde{A}(x) \tag{26}$$

$$\frac{\partial \tilde{Q}(t,x)}{\partial t} = -\tau^2 \frac{\partial^2 \tilde{Q}(t,x)}{\partial x^2} - Q^*\, f_{KK}(K^*)\, \tilde{K}(t,x) - Q^*\, f_{KA}(K^*)\, \tilde{A}(x). \tag{27}$$

Using finite Fourier transforms for the location variable, transforms this PDE system into an ODE system,

$$\begin{pmatrix} \mathcal{K}'_\omega(t) \\ \mathcal{Q}'_\omega(t) \end{pmatrix} = \mathbf{J}_\omega \begin{pmatrix} \mathcal{K}_\omega(t) \\ \mathcal{Q}_\omega(t) \end{pmatrix} + \mathbf{J}_{A,\omega}\, \mathcal{A}_\omega \text{ where } \mathbf{J}_{A,\omega} \equiv \begin{pmatrix} f_A(K^*) \\ -Q^*\, f_{KA}(K^*) \end{pmatrix}, \text{ for each } \omega \in \mathbb{Z}$$

where $\mathbf{J}_\omega$ is in equation (21) and $\mathcal{A}_\omega = \int_{-1/2}^{1/2} A(x)\, e^{-i2\pi\omega x}\, dx$. Using a similar approach as in the proof of Proposition 4 to find the solution along the generalized stable manifold, and setting $\mathcal{K}_\omega(0) = 0$, because the productivity shock is unanticipated and the economies start at a flat steady-state, yields

$$\begin{pmatrix} \mathcal{K}_\omega(t) \\ \mathcal{Q}_\omega(t) \end{pmatrix} = \begin{pmatrix} \bar{\mathcal{K}}_\omega \\ \bar{\mathcal{Q}}_\omega \end{pmatrix} - \bar{\mathcal{K}}_\omega \begin{pmatrix} 1 \\ \psi^-_\omega \end{pmatrix} e^{\lambda^-_\omega\, t}, \text{ for } t \in \mathrm{T},$$

where ψ_ω^- is in equation (23). In this solution, the finite Fourier transforms of the long run multipliers are

$$\begin{pmatrix} \bar{\mathcal{K}}_\omega \\ \bar{\mathcal{Q}}_\omega \end{pmatrix} = -\mathbf{J}_\omega^{-1}\,\mathbf{J}_{A,\omega}\,\mathcal{A}_\omega = \begin{pmatrix} \mathcal{M}_\omega^k \\ \mathcal{M}_\omega^q \end{pmatrix} \mathcal{A}_\omega, \text{ for each } \omega \in \mathbb{Z},$$

where

$$\mathcal{M}_\omega^k \equiv -\frac{\xi_\omega\, f_A(K^*) - \frac{Q^*}{u''(C^*)}\, f_{AK}(K^*)}{\xi_\omega\,(\rho - \xi_\omega) - \mu} \tag{28}$$

$$\mathcal{M}_\omega^q \equiv -\frac{Q^*\left(f_A(K^*)\, f_{KK}(K^*) + (\xi_\omega - \rho) f_{AK}(K^*)\right)}{\xi_\omega\,(\rho - \xi_\omega) - \mu} \tag{29}$$

Using finite Fourier inversion we get the multipliers in the original location variables

$$\tilde{K}(t,x) = \sum_{\omega \in \mathbb{Z}} \bar{K}_\omega \left(1 - e^{\lambda_\omega^-\, t}\right) e^{i2\pi\omega x}, \text{ for } (t,x) \in \mathrm{T} \times \mathrm{X} \tag{30}$$

$$\tilde{Q}(t,x) = \sum_{\omega \in \mathbb{Z}} \left(\bar{Q}_\omega - \psi_\omega^-\, \bar{K}_\omega\, e^{\lambda_\omega^-\, t}\right) e^{i2\pi\omega x}, \text{ for } (t,x) \in \mathrm{T} \times \mathrm{X} \tag{31}$$

Given the boundedness of the functions involved and of the set X, we can have a singular integral representations of the multipliers, for every $t \in \mathrm{T}$ as in equations (52) and (53). □

Appendix References

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